A New Encoding and Implementation of Not Necessarily Closed Convex Polyhedra*

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Abstract. Convex polyhedra, commonly employed for the analysis and verification of both hardware and software, may be defined either by a finite set of linear inequality constraints or by finite sets of generating points and rays of the polyhedron. Although most implementations of the polyhedral operations assume that the polyhedra are topologically closed (i.e., all the constraints defining them are non-strict), several analyzers and verifiers need to compute on a domain of convex polyhedra that are not necessarily closed (NNC). The usual approach to implementing NNC polyhedra is to embed them into closed polyhedra in a vector space having one extra dimension and reuse the tools and techniques already available for closed polyhedra. Previously, this embedding has been designed so that a constant number of constraints and a linear number of generators have to be added to the original NNC specification of the polyhedron. In this paper we explore an alternative approach: while still using an extra dimension to represent the NNC polyhedron by a closed polyhedron, the new embedding adds a linear number of constraints and a constant number of generators. As far as the issue of providing a non-redundant description of the NNC polyhedron is concerned, we generalize the results established in a previous paper so that they apply to both encodings.

1 Introduction

Many applications of static analysis and verification compute on some abstract domain based on convex polyhedra [5]. Traditionally, most of these applications are restricted to convex polyhedra that are topologically closed. When adopting the *Double Description* (DD) method [10], a closed convex polyhedron can be specified in two ways, using a *constraint system* or a *generator system*: the constraint system contains a finite set of linear non-strict inequality constraints; the generator system contains two finite sets of vectors, collectively called *generators*, which are rays and points of the polyhedron.

* This work has been partly supported by MURST project "Aggregate- and number-reasoning for computing: from decision algorithms to constraint programming with multisets, sets, and maps".

Some applications of static analysis and verification, including recent proposals such as [4], need to compute on the domain of *not necessarily closed* (NNC) convex polyhedra. By definition, any NNC polyhedron can be represented by a so-called *mixed constraint system*, that is, a constraint system where a further finite set of linear *strict* inequality constraints is allowed to occur. The usual approach for implementing NNC polyhedra is to embed them into closed polyhedra in a vector space with one extra dimension. While this idea, originally proposed in [7] and also described in [8], proved to be quite effective, its direct application results in a low-level user interface where most of the geometric intuition of the DD method gets lost under the "implementation details".[3]

A much cleaner approach was proposed in [1, 2], where the concept of generator of an NNC polyhedron is extended to also account for the *closure points* of the polyhedron. In particular, it is shown that any NNC polyhedron can be defined directly by means of an *extended generator system*, namely, a triple of finite sets containing rays, points and closure points of the polyhedron. By combining the mixed constraint systems with these extended generator systems for describing NNC polyhedra we can obtain a two-fold improvement over the proposal in [7, 8]: easier generalizations and a natural, implementation-independent interface.

Easier generalizations. Several complex operators, whose definition is in terms of the rays and points of the standard generator systems for closed polyhedra, need to be generalized to NNC polyhedra. Examples are given by the *time-elapse* operator of [7, 8] and the generators-based widening of [3]. The notion of extended generator system proved to be very effective in the definition and justification of these generalizations. As an example, let us consider a very basic operator: the inclusion test between two polyhedra. The usual implementation for closed polyhedra is based on the following specification in terms of their constraint and generator systems. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be closed polyhedra such that $\mathcal{P}_1$ is defined by the generator system $\mathcal{G}_1$ and $\mathcal{P}_2$ by the constraint system $\mathcal{C}_2$. Then we have $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if all the generators in $\mathcal{G}_1$ *satisfy* all the constraints in $\mathcal{C}_2$. In order to test whether or not a generator $\boldsymbol{g}$ satisfies a constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$, it is sufficient to determine if the scalar product $s = \langle \boldsymbol{a}, \boldsymbol{g} \rangle$ is such that $s \geq b$, when $\boldsymbol{g}$ is a point, or such that $s \geq 0$, when $\boldsymbol{g}$ is a ray. Consider now the generalization to two polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$ that are not necessarily closed. With the high-level interface proposed in [2], the inclusion test can be easily specified using *the same approach* described above: we only need to generalize the case analysis of the satisfaction test to also cover the combinations provided by the additional constraint and generator types (i.e., strict inequalities and closure points), as shown in Table 1. The elegance of this generalization is better appreciated if contrasted with the specification of the inclusion test on the low-level implementation of [7], informally described in the same paper, which appears to be much more tricky

[3] This has a direct, negative impact on the usability of the resulting software: on this subject, see [2, Section 4.1, page 218], [6, Section 4.5, pp. 10–11], and [9, Section 1.1.4, page 10].

and obscure. The reason is that in [7] the reader has no high-level interpretation of the generators occurring in the low-level encoding.

	Generator type		
Constraint type	ray	point	closure point
non-strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$
strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle > b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$

Table 1. Checking whether a constraint is satisfied by a generator.

A natural, implementation-independent interface. The combination of mixed constraint systems and extended generator systems offers another improvement over the proposal in [7, 8]: a high-level user interface that is completely separate from the implementation. On the one hand, an NNC polyhedron can be presented to the client application directly in terms of its defining strict and non-strict constraints or its generating rays, points and closure points; there is no need for the client to be aware of the use of an additional space dimension in the implementation and all issues related to its correct handling, such the strong minimization procedures [2]. On the other hand, by relying on the high-level specification only, the client application will be unaffected by the wider adoption of lazy and incremental computation techniques in the procedures implementing the operators on convex polyhedra. Moreover, if all the functionalities and invariants of the interface are maintained, it is then possible to change the low-level data structures without affecting the application.

In this paper we exploit the latter possibility by introducing an alternative class of closed polyhedra for implementing the NNC polyhedra. The basis of this representation is a simple generalization of the class of polyhedra used in [7, 8] and also in [2]. The new class continues to employ an additional dimension to encode whether or not each affine half-space defining the NNC polyhedron is closed and relies on the same semantic function given in [2] for extracting the NNC polyhedron it embeds. We describe two alternative specializations of this class for representing the NNC polyhedra. One of these, shown to be biased for the use of the constraint representation, corresponds to the embedding defined in [2] while the other, which is biased for the use of the generator representation, is new to this paper. Moreover, we generalize the notion of strong minimal form [2] so that it is applicable to all the above classes of closed polyhedra.

One interesting and potentially useful consequence of having the option of these alternative implementations is that, depending on the number of strict constraints in the constraint system compared with the number of closure points that are also points in the generator system, the choice of representation will affect the efficiency of the polyhedral operations. The *Parma Polyhedra Library*[4],

[4] Publicly available at URI `http://www.cs./unipr.it/ppl/`. The implementation described in this paper is available in the `alt_nnc` branch of the PPL's CVS repository.

a modern C++ library for the manipulation of convex polyhedra, has been extended so as to implement both approaches, so that it will be possible to perform experiments to compare their efficiencies.

The paper is structured as follows: Section 2 recalls the required concepts and notations; Section 3 presents a general class and two special subclasses of the set of closed polyhedra that are appropriate for the representation of NNC polyhedra; Section 4 generalizes, to all the above classes of closed polyhedra, the notion of strong minimal form introduced in [2]; Section 5 concludes.

2 Preliminaries

We first define some necessary terminology and notation.

The set of non-negative reals is denoted by $\mathbb{R}_+$. In the paper, all topological arguments refer to the Euclidean topological space $\mathbb{R}^n$, for $n \in \mathbb{N}$. If $S \subseteq \mathbb{R}^n$, then the *topological closure* $\mathbb{C}(S)$ is defined as $\bigcap\{\, C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed} \,\}$.

For each $i \in \{1, \dots, n\}$, v_i denotes the i-th component of the (column) vector $\boldsymbol{v} \in \mathbb{R}^n$. We denote by $\boldsymbol{0}$ the vector of $\mathbb{R}^n$ having all components equal to zero. A vector $\boldsymbol{v} \in \mathbb{R}^n$ can also be interpreted as a matrix in $\mathbb{R}^{n \times 1}$ and manipulated accordingly with the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, which is denoted by $\boldsymbol{v}^{\mathrm{T}}$. The *scalar product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, denoted $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$, is the real number $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \sum_{i=1}^{n} v_i w_i$.

For any relational operator $\bowtie \in \{=, \geq, \leq, <, >\}$, we write $\boldsymbol{v} \bowtie \boldsymbol{w}$ to denote the conjunctive proposition $\bigwedge_{i=1}^{n} (v_i \bowtie w_i)$. In contrast, $\boldsymbol{v} \neq \boldsymbol{w}$ will denote the proposition $\neg(\boldsymbol{v} = \boldsymbol{w})$. For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \boldsymbol{0}$, the linear inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$) defines a topologically closed (resp., open) affine half-space of $\mathbb{R}^n$. We do not distinguish between syntactically different constraints defining the same affine half-space so that, e.g., $x \geq 2$ and $2x \geq 4$ are considered to be the same constraint.

A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called a *closed polyhedron* if either $\mathcal{P}$ can be expressed as the intersection of a finite number of closed affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all closed polyhedra on $\mathbb{R}^n$ is denoted by $\mathbb{CP}_n$. A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called an *NNC polyhedron* if either $\mathcal{P}$ can be expressed as the intersection of a finite number of (not necessarily closed) affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all NNC polyhedra on $\mathbb{R}^n$ is denoted by $\mathbb{P}_n$. The set $\mathbb{P}_n$, when partially ordered by subset inclusion, is a lattice and $\mathbb{CP}_n$ is a sublattice of $\mathbb{P}_n$ (note that $\mathbb{CP}_n = \mathbb{P}_n$ if and only if $n = 0$). The binary meet operation is given by set-intersection, whereas the binary join operation, denoted $\uplus$, is called *convex polyhedral hull*, *poly-hull* for short. In this paper, we only consider polyhedra in $\mathbb{P}_n$ when $n > 0$.

A *mixed constraint system* $\mathcal{C}$ is a finite set of linear inequality constraints and we write $\mathrm{con}(\mathcal{C})$ to denote the polyhedron described by $\mathcal{C}$.

Suppose that $\mathcal{P} \in \mathbb{P}_n$ is non-empty. A vector $\boldsymbol{p} \in \mathbb{R}^n$ is a *point* of $\mathcal{P}$ if $\boldsymbol{p} \in \mathcal{P}$; a vector $\boldsymbol{r} \in \mathbb{R}^n$ such that $\boldsymbol{r} \neq \boldsymbol{0}$ is a *ray* of $\mathcal{P}$ if, for every point $\boldsymbol{p} \in \mathcal{P}$ and every $\rho \in \mathbb{R}_+$, we have $\boldsymbol{p} + \rho\boldsymbol{r} \in \mathcal{P}$; a vector $\boldsymbol{c} \in \mathbb{R}^n$ is a *closure point* of $\mathcal{P}$ if $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$. Given three finite sets of vectors $R, P, C \subseteq \mathbb{R}^n$, where $R = \{\boldsymbol{r}_1, \dots, \boldsymbol{r}_r\}$ and

$\mathbf{0} \notin R$, $P = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_p\}$ and $C = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_c\}$, then the triple $\mathcal{G} = (R, P, C)$ is called an *extended generator system* [2] for the NNC polyhedron

$$\mathrm{gen}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \left\{ \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i \;\middle|\; \begin{array}{l} \boldsymbol{\rho} \in \mathbb{R}_+^r, \boldsymbol{\pi} \in \mathbb{R}_+^p, \boldsymbol{\gamma} \in \mathbb{R}_+^c, \\ \boldsymbol{\pi} \neq \mathbf{0}, \sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1 \end{array} \right\}.$$

The polyhedron $\mathrm{gen}(\mathcal{G})$ is empty if and only if $P = \varnothing$. For a non-empty polyhedron $\mathcal{P}$, vectors in R, P, and C are rays, points and closure points of $\mathcal{P}$, respectively. We define an ordering $\sqsubseteq$ on extended generator systems such that, for any generator systems $\mathcal{G}_1 = (R_1, P_1, C_1)$ and $\mathcal{G}_2 = (R_2, P_2, C_2)$, $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ if and only if $R_1 \subseteq R_2$, $P_1 \subseteq P_2$ and $C_1 \subseteq C_2$; if, in addition, $\mathcal{G}_1 \neq \mathcal{G}_2$, we write $\mathcal{G}_1 \sqsubset \mathcal{G}_2$. When $C = \varnothing$, we will omit it from the generator system and simply write $\mathcal{G} = (R, P)$. In this case, the system $\mathcal{G}$ that defines the closed polyhedron $\mathcal{P} = \mathrm{gen}(\mathcal{G})$, is called a *(standard) generator system* for $\mathcal{P}$.

Consider a mixed constraint system $\mathcal{C}$, an extended generator system $\mathcal{G}$, and a polyhedron $\mathcal{P}$. If $\mathrm{con}(\mathcal{C}) = \mathrm{gen}(\mathcal{G}) = \mathcal{P}$, then $(\mathcal{C}, \mathcal{G})$ is said to be a *DD pair* for $\mathcal{P}$, and we write $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. We say that

- $\mathcal{C}$ is in *minimal form* if there does not exist $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con}(\mathcal{C}') = \mathcal{P}$;
- $\mathcal{G}$ is in *minimal form* if there does not exist $\mathcal{G}' \sqsubset \mathcal{G}$ such that $\mathrm{con}(\mathcal{G}') = \mathcal{P}$;
- the DD pair $(\mathcal{C}, \mathcal{G})$ is in *minimal form* if $\mathcal{C}$ and $\mathcal{G}$ are both in minimal form.

3 Representing NNC Polyhedra

The idea underlying the proposal of [7, 8] is to encode each NNC polyhedron of $\mathbb{P}_n$ into a closed polyhedron of $\mathbb{CP}_{n+1}$. In the following, we denote by ϵ the variable corresponding to the $(n+1)$-st Cartesian axis of $\mathbb{R}^{n+1}$. The interpretation function $[\![\cdot]\!] \colon \mathbb{CP}_{n+1} \to \mathbb{P}_n$ maps any closed polyhedron in $\mathbb{CP}_{n+1}$ to an NNC polyhedron in $\mathbb{P}_n$; in particular, points in the closed polyhedron with a positive ϵ-coordinate correspond to points in the NNC polyhedron.

Definition 1. (Represented NNC polyhedron.) *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be a closed polyhedron. $\mathcal{R}$ is said to* represent *the NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ if and only if*

$$\mathcal{P} = [\![\mathcal{R}]\!] \stackrel{\mathrm{def}}{=} \left\{ \boldsymbol{v} \in \mathbb{R}^n \;\middle|\; \exists e \in \mathbb{R} \,.\, \bigl(e > 0 \wedge (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}\bigr) \right\}. \tag{1}$$

Note that any closed polyhedron that is included in the half-space defined by the constraint $\epsilon \leq 0$ actually represents the empty NNC polyhedron.

Not all the polyhedra in $\mathbb{CP}_{n+1}$ are good candidates for representing an NNC polyhedron in $\mathbb{P}_n$. The rationale driving the choice of an appropriate subclass of $\mathbb{CP}_{n+1}$ is that most of the operators defined on the domain of closed polyhedra could be used, with no more than minor modifications, to implement corresponding operators on the domain of NNC polyhedra. For instance, one would like to implement the intersection and the poly-hull of two NNC polyhedra by computing the intersection and the poly-hull of their closed representations, respectively. Under such a requirement, we will define two alternative representations for NNC polyhedra. The two classes of closed polyhedra used for these representations are instances of a more general class of closed polyhedra.

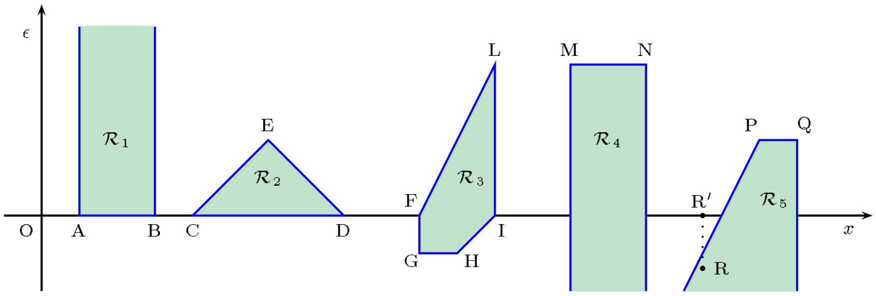

Fig. 1. Only $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ are ϵ-polyhedra.

Definition 2. **(ϵ-polyhedron.)** *A closed polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be an* ϵ-polyhedron *if and only if*

$$\exists \delta \in \mathbb{R} \,.\, \Bigl(\delta > 0 \land \mathcal{R} \subseteq \mathrm{con}\bigl(\{\epsilon \le \delta\}\bigr)\Bigr); \tag{2}$$

$$\forall \boldsymbol{v} \in \mathbb{R}^n, e \in \mathbb{R} : (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R} \implies (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}. \tag{3}$$

The polyhedron $\mathcal{R}$ is said to be an ϵ-polyhedron for $\mathcal{P} \in \mathbb{P}_n$, *denoted $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, if $\mathcal{R}$ is an ϵ-polyhedron and $\mathcal{P} = [\![\mathcal{R}]\!]$.*

Condition (3) that every point in the ϵ-polyhedron $\mathcal{R}$ has a projection on the hyperplane defined by the constraint $(\epsilon = 0)$ corresponds to a dual property concerning the constraints for $\mathcal{R}$.

Proposition 1. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \subseteq \mathrm{con}\bigl(\{\epsilon \le \delta\}\bigr)$, where $\delta > 0$. Then $\mathcal{R}$ is an ϵ-polyhedron if and only if*

$$\mathcal{R} \subseteq \mathrm{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \ge b\bigr\}\Bigr) \implies \mathcal{R} \subseteq \mathrm{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \ge b\bigr\}\Bigr). \tag{4}$$

In Figure 1 we show several examples of polyhedra in $\mathbb{CP}_2$ (representing NNC polyhedra in $\mathbb{P}_1$), a subset of which happens to be ϵ-polyhedra. In particular, the semi-column polyhedron $\mathcal{R}_1$, which according to Definition 1 represents the closed interval $\mathcal{P}_1 = \mathrm{con}\bigl(\{1 \le x \le 3\}\bigr)$, is not an ϵ-polyhedron, because it is not provided with a finite upper-bound on the ϵ coordinate, therefore violating condition (2) of Definition 2. The triangle $\mathcal{R}_2$ is an ϵ-polyhedron for the open segment $\mathcal{P}_2 = \mathrm{con}\bigl(\{4 < x < 8\}\bigr)$. Polyhedron $\mathcal{R}_3$ is an ϵ-polyhedron for the segment $\mathcal{P}_3 = \mathrm{con}\bigl(\{10 < x \le 12\}\bigr)$, which is neither closed nor open. Similarly, $\mathcal{R}_4$ is an ϵ-polyhedron for the closed segment $\mathcal{P}_4 = \mathrm{con}\bigl(\{14 \le x \le 16\}\bigr)$. Finally, polyhedron $\mathcal{R}_5$ represents the NNC polyhedron $\mathcal{P}_5 = \mathrm{con}\bigl(\{18 \le x \le 20\}\bigr)$, but it is not an ϵ-polyhedron because it violates condition (3) of Definition 2. For instance, even though $\mathrm{R} \in \mathcal{R}_5$, we have $\mathrm{R}' \notin \mathcal{R}_5$.

If we are to provide an implementation-independent interface for the user, we need to be able to extract from the constraint and generator systems describing an ϵ-polyhedron, the corresponding mixed constraint system and extended generator system describing the NNC polyhedron it represents. Reasoning at the intuitive level, consider an arbitrary ϵ-polyhedron, such as $\mathcal{R}_3$ in Figure 1. Then, it is worth noting that any facet that is parallel to the ϵ axis, such as the segment $[\mathrm{I}, \mathrm{L}]$, corresponding to an inequality constraint having a zero coefficient for the ϵ variable, will encode a *non-strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x \leq 12$). On the other hand, any facet such as the segment $[\mathrm{L}, \mathrm{F}]$, corresponding to an inequality constraint having a negative coefficient for the ϵ variable, will encode a *strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x > 10$). Equivalently, we could have noted that in polyhedron $\mathcal{R}_3$ the points having a strictly positive ϵ coordinate can be chosen arbitrarily close to vertex $\mathrm{F} = (10, 0)^{\mathrm{T}}$, but all the points having value 10 for their x coordinate happen to have a non-positive ϵ coordinate. Thus, the vector $\mathrm{F}' = (10) \in \mathbb{R}^1$ represented by F is not a point of the NNC polyhedron $\mathcal{P}_3$, but it is one of its closure points. All of the above observations can be formalized as follows.

Definition 3. (Encoded descriptions.) *Let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ *be a DD pair for a closed polyhedron. Then, if* $[\![\mathcal{R}]\!] \neq \varnothing$, *the* mixed constraint system encoded by $\mathcal{C}$ *is defined as* $\mathrm{con\_enc}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \mathcal{C}_S \cup \mathcal{C}_{NS}$, *where*

$$\mathcal{C}_S \stackrel{\mathrm{def}}{=} \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle > b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}, \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \Bigr\},$$
$$\mathcal{C}_{NS} \stackrel{\mathrm{def}}{=} \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}, \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \notin \mathcal{C}_S \Bigr\}.$$

If $[\![\mathcal{R}]\!] = \varnothing$, *then we define* $\mathrm{con\_enc}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \{x_1 > 0, -x_1 > 0\}$. *Also, the* extended generator system encoded by $\mathcal{G} = (R, P)$ *is defined as* $\mathrm{gen\_enc}(\mathcal{G}) \stackrel{\mathrm{def}}{=} (R', P', C')$, *where*

$$R' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{r} \bigm| (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R \bigr\},$$
$$P' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{p} \bigm| (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in P, e > 0 \bigr\},$$
$$C' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{c} \bigm| (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P, \boldsymbol{c} \notin P' \bigr\}.$$

The following proposition states the correctness of the two mappings introduced above.

Proposition 2. *Let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ *be an* ϵ*-polyhedron. Then*

$$[\![\mathcal{R}]\!] = \mathrm{con}\bigl(\mathrm{con\_enc}(\mathcal{C})\bigr) = \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G})\bigr). \tag{5}$$

Figure 2, which shows the poly-hulls of some of the polyhedra in Figure 1, provides a graphical and informal justification for the two conditions stated in Definition 2. Suppose we do not enforce condition (2) of Definition 2, thus admitting polyhedra such as $\mathcal{R}_1$, and consider the poly-hull $\mathcal{P}_1 \uplus \mathcal{P}_2 = \mathrm{con}\bigl(\{1 \leq x < 8\}\bigr)$. The poly-hull $\mathcal{R}_1 \uplus \mathcal{R}_2$ of the two encodings for $\mathcal{P}_1$ and $\mathcal{P}_2$ represents a wrong result, since $[\![\mathcal{R}_1 \uplus \mathcal{R}_2]\!] = \mathrm{con}\bigl(\{1 \leq x \leq 8\}\bigr)$. Suppose now we do not enforce condition (3) of Definition 2, thus allowing for polyhedra such as $\mathcal{R}_5$, and

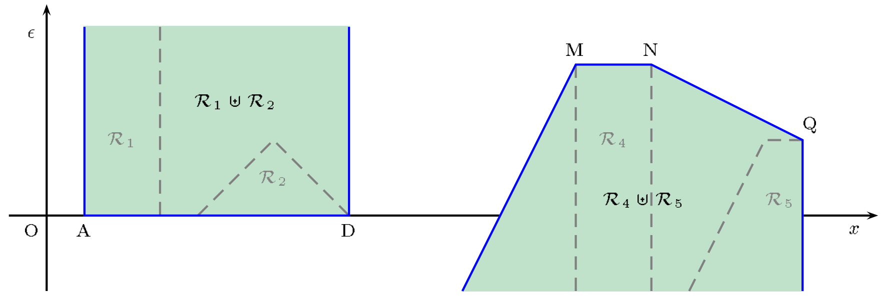

Fig. 2. $\mathcal{R}_1 \uplus \mathcal{R}_2$ does not represent the NNC polyhedron $\mathcal{P}_1 \uplus \mathcal{P}_2$; similarly, $\mathcal{R}_4 \uplus \mathcal{R}_5$ does not represent the NNC polyhedron $\mathcal{P}_4 \uplus \mathcal{P}_5$.

consider the poly-hull $\mathcal{P}_4 \uplus \mathcal{P}_5 = \mathrm{con}\bigl(\{14 \leq x \leq 20\}\bigr)$. Again, the computation of this poly-hull using the closed encodings of its arguments provides a wrong result, since we have $[\![\mathcal{R}_4 \uplus \mathcal{R}_5]\!] = \mathrm{con}\bigl(\{12 < x \leq 20\}\bigr)$.

We now consider two special subclasses of the class of ϵ-polyhedra. The first of these requires zero as a lower bound for the ϵ dimension.

Definition 4. (C-ϵ-polyhedron.) *An ϵ-polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be* constraint-biased *and called a* C-ϵ-polyhedron *if and only if*

$$\mathcal{R} \subseteq \mathrm{con}\bigl(\{\epsilon \geq 0\}\bigr).$$

We write $\mathcal{R} \Rrightarrow_C \mathcal{P}$ if $\mathcal{R}$ is a C-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

The set of constraint-biased ϵ-polyhedra corresponds, essentially, to the class of polyhedra originally proposed in [7, 8]. This is also the same class considered in [2], where these polyhedra were called ϵ-representations. Thus many of the definitions and results below concerning C-ϵ-polyhedra and the embedding of the NNC polyhedra in them are taken from [2].

In [2], we have shown how a C-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ may be constructed directly from the constraint and generator systems for $\mathcal{P}$.

Definition 5. ($\mathrm{con\_repr}_C$ and $\mathrm{gen\_repr}_C$.) *Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The* constraint-biased representation *of $\mathcal{C}$ is the constraint system $\mathrm{con\_repr}_C(\mathcal{C})$ on the vector space $\mathbb{R}^{n+1}$ where*

$$\begin{aligned}
\mathrm{con_repr}_C(\mathcal{C}) \stackrel{\mathrm{def}}{=} \{0 \leq \epsilon \leq 1\} & \\
\cup \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b \Bigm| & \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \in \mathcal{C} \Bigr\} \\
\cup \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b \Bigm| & \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b\bigr) \in \mathcal{C} \Bigr\}.
\end{aligned}$$

The constraint-biased representation *of $\mathcal{G} = (R, P, C)$ is the generator system $\mathrm{gen\_repr}_C(\mathcal{G}) = (R', P')$ on the vector space $\mathbb{R}^{n+1}$ where*

$$\begin{aligned}
R' &\stackrel{\mathrm{def}}{=} \bigl\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| \boldsymbol{r} \in R \bigr\}, \\
P' &\stackrel{\mathrm{def}}{=} \bigl\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \bigm| \boldsymbol{p} \in P \bigr\} \cup \bigl\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| \boldsymbol{q} \in P \cup C \bigr\}.
\end{aligned}$$

Observe that, in the mapping defined by the representation function $\mathrm{gen\_repr}_C$ and using the notation in Definition 5, each point in P corresponds to two distinct points in P', having a positive and a zero ϵ coordinate, respectively. This ensures that condition (3) of Definition 2 is met. In general, the above encodings require a constant number of additional constraints versus a linear number of additional generators: this is the reason why ϵ-polyhedra in this subclass are called "constraint-biased".

The second special subclass of ϵ-polyhedra requires that all the non-empty ϵ-polyhedra have the ray $-\boldsymbol{e}_\epsilon \stackrel{\mathrm{def}}{=} (\boldsymbol{0}^\mathrm{T}, -1)^\mathrm{T}$ so that there is no lower bound for the ϵ dimension.

Definition 6. (G-ϵ-polyhedron.) *An ϵ-polyhedron $\mathcal{R} = \mathrm{gen}\bigl((R, P)\bigr) \in \mathbb{CP}_{n+1}$ is said to be* generator-biased *and called a* G-ϵ-polyhedron *if and only if*

$$\mathcal{R} \supseteq \mathrm{gen}\Bigl(\bigl(\{-\boldsymbol{e}_\epsilon\}, P\bigr)\Bigr).$$

We write $\mathcal{R} \Rrightarrow_G \mathcal{P}$ if $\mathcal{R}$ is a G-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

As for the constraint-biased case, generator-biased ϵ-polyhedra can also be used for representing any NNC polyhedron. In particular, a G-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ may be constructed directly from the constraint and generator systems for $\mathcal{P}$.

Definition 7. ($\mathrm{con\_repr}_G$ and $\mathrm{gen\_repr}_G$.) *Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The* generator-biased representation *of $\mathcal{C}$ is the constraint system $\mathrm{con\_repr}_G(\mathcal{C})$ on the vector space $\mathbb{R}^{n+1}$ where*

$$\begin{aligned}
\mathrm{con_repr}_G(\mathcal{C}) \stackrel{\mathrm{def}}{=} \bigl\{0 \leq \epsilon \leq 1\bigr\} & \\
\cup \Bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle - 1 \cdot \epsilon \geq b \Bigm| & \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}\Bigr\} \\
\cup \Bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b \Bigm| & \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}\Bigr\} \\
\cup \Bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b \Bigm| & \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geq b\bigr) \in \mathcal{C}\Bigr\}.
\end{aligned}$$

The generator-biased representation *of $\mathcal{G} = (R, P, C)$ is the generator system $\mathrm{gen\_repr}_G(\mathcal{G}) = (R', P')$ on the vector space $\mathbb{R}^{n+1}$ where*

$$\begin{aligned}
R' &= \bigl\{-\boldsymbol{e}_\epsilon\bigr\} \cup \bigl\{\, (\boldsymbol{r}^\mathrm{T}, 0)^\mathrm{T} \bigm| \boldsymbol{r} \in R \,\bigr\}, \\
P' &= \bigl\{\, (\boldsymbol{p}^\mathrm{T}, 1)^\mathrm{T} \bigm| \boldsymbol{p} \in P \,\bigr\} \cup \bigl\{\, (\boldsymbol{q}^\mathrm{T}, 0)^\mathrm{T} \bigm| \boldsymbol{q} \in C \,\bigr\}.
\end{aligned}$$

It can be seen that, for each strict inequality contained in $\mathcal{C}$, the representation function $\mathrm{con\_repr}_G$ adds both the strict and the non-strict inequality encodings. This is similar to what is done for points in Definition 5 and, by virtue of Proposition 1, ensures that condition (3) of Definition 2 is met. In contrast, for each point in the generator system, the function $\mathrm{gen\_repr}_G$ does not add the corresponding closure point. In fact, these closure points are not needed, because they can be generated by combining the corresponding point with the ray $-\boldsymbol{e}_\epsilon$, which is always added. Since the encodings for ϵ-polyhedra in this subclass require a linear number of additional constraints versus a constant number of additional generators, they are called “generator-biased”.

Returning to Figure 1, it can be observed that $\mathcal{R}_2$ is a constraint-biased ϵ-polyhedron, $\mathcal{R}_4$ is a generator-biased ϵ-polyhedron, whereas the ϵ-polyhedron $\mathcal{R}_3$ is neither constraint-biased nor generator-biased. By comparing $\mathcal{R}_3$ with $\mathcal{R}_1$ and $\mathcal{R}_2$ it can be seen that those ϵ-polyhedra that are not members of one of the two subclasses can require *both* a linear number of additional constraints and a linear number of additional generators (with respect to the original NNC descriptions), resulting in a significant waste of both memory space and computational time.

The following result formalizes the correctness of the encodings introduced in Definitions 5 and 7.

Proposition 3. *Let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. *Then*

1. $\mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C})\bigr) \Rrightarrow_C \mathcal{P}$, $\quad \mathrm{con}\bigl(\mathrm{con\_repr}_G(\mathcal{C})\bigr) \Rrightarrow_G \mathcal{P}$;
2. $\mathrm{gen}\bigl(\mathrm{gen\_repr}_C(\mathcal{G})\bigr) \Rrightarrow_C \mathcal{P}$, $\quad \mathrm{gen}\bigl(\mathrm{gen\_repr}_G(\mathcal{G})\bigr) \Rrightarrow_G \mathcal{P}$.

The next proposition shows that most of the operators defined on the domain of NNC polyhedra $\mathbb{P}_n$ can be mapped into the corresponding operators on the class of ϵ-polyhedra defined on $\mathbb{CP}_{n+1}$.

Proposition 4. *Let* $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. *Suppose* $\mathcal{R} \Rrightarrow_Y \mathcal{P}$, *and* $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1$ *and* $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2$. *Then*

1. $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \cap \mathcal{P}_2$;
2. $(\mathcal{P}_1 \neq \varnothing \wedge \mathcal{P}_2 \neq \varnothing) \implies (\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \uplus \mathcal{P}_2)$;
3. *let* $f \stackrel{\mathrm{def}}{=} \lambda \boldsymbol{x} \in \mathbb{R}^n \,.\, A\boldsymbol{x} + \boldsymbol{b}$ *be any affine transformation defined on* $\mathbb{P}_n$; *then* $g(\mathcal{R}) \Rrightarrow_Y f(\mathcal{P})$, *where*

$$g \stackrel{\mathrm{def}}{=} \lambda \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} \in \mathbb{R}^{n+1} \,.\, \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix}$$

   is the corresponding affine transformation on $\mathbb{CP}_{n+1}$.

Hence, operations such as the intersection of NNC polyhedra and the application of affine transformations can be safely performed on any ϵ-polyhedra for the arguments; the same is true for the convex polyhedral hull operation, provided neither of the arguments is empty. Moreover, both the constraint-biased and the generator-biased subclasses are closed under the application of these operators.

4 Strong Minimization of ϵ-Polyhedra

As pointed out in [2], the usual minimization of the descriptions of a (constraint-biased) ϵ-polyhedron does not enforce the minimization of the encoded descriptions for the represented NNC polyhedron. The solution proposed in [2] is the definition of a stronger form of minimization to be applied to the descriptions of constraint-biased ϵ-polyhedra. We here define the same notion of strong minimal form, but this time for arbitrary ϵ-polyhedra, being careful that if it is constraint- or generator-biased before minimization, then it remains constraint- or generator-biased, respectively, after the minimization.

Definition 8. (Strong minimal form for ϵ-polyhedra.) *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then*

- $\mathcal{C}$ *is in* strong minimal form *if there does not exist a constraint system $\mathcal{C}' \subset \mathcal{C}$ such that* $\mathrm{con}\bigl(\mathcal{C}' \cup \{\epsilon \leq 1\}\bigr) \Rrightarrow_\epsilon \mathcal{P}$ *and* $\mathrm{con\_enc}(\mathcal{C}') \subset \mathrm{con\_enc}(\mathcal{C})$*;*
- $\mathcal{G}$ *is in* strong minimal form *if there does not exist a generator system $\mathcal{G}' \sqsubset \mathcal{G}$ such that* $\mathrm{gen}(\mathcal{G}') \Rrightarrow_\epsilon \mathcal{P}$ *and* $\mathrm{gen\_enc}(\mathcal{G}') \sqsubset \mathrm{gen\_enc}(\mathcal{G})$.

The computation of strong minimal forms (smf's, for short) requires the removal of non-essential constraints and generators, whose efficient detection is based on the checking of particular saturation conditions. The following notation is needed for a formal definition of these conditions.

Let $\mathcal{R} = \mathrm{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$. The set of *strict* and *non-strict inequality encodings* $\mathcal{C}_>$ and $\mathcal{C}_\geq$ of $\mathcal{C}$ are defined as

$$\mathcal{C}_> \stackrel{\mathrm{def}}{=} \Bigl\{\, \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C} \Bigm| \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \,\Bigr\};$$
$$\mathcal{C}_\geq \stackrel{\mathrm{def}}{=} \Bigl\{\, \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C} \Bigm| \boldsymbol{a} \neq \boldsymbol{0}, s = 0 \,\Bigr\}.$$

We say that a constraint $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$ is *unmatched* in $\mathcal{C}$ if $s < 0$ and $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b\bigr) \notin \mathcal{C}$.

The sets of *point encodings* $\mathcal{G}_P \subseteq P$, *closure point encodings* $\mathcal{G}_C \subseteq P$, and *ray encodings* $\mathcal{G}_R \subseteq R$ of the generator system $\mathcal{G} = (R, P)$ are defined as follows:

$$\mathcal{G}_P \stackrel{\mathrm{def}}{=} \bigl\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e > 0 \,\bigr\};$$
$$\mathcal{G}_C \stackrel{\mathrm{def}}{=} \bigl\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e = 0 \,\bigr\};$$
$$\mathcal{G}_R \stackrel{\mathrm{def}}{=} \bigl\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \bigm| e = 0 \,\bigr\}.$$

A point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P$ is said to be *unmatched* in $\mathcal{G}$ if $e > 0$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \notin P$.

We say that a point $\boldsymbol{p}$ (resp., a ray $\boldsymbol{r}$) *saturates* a constraint $\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b$ if and only if $\langle \boldsymbol{a}, \boldsymbol{p}\rangle = b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{r}\rangle = 0$). For any point $\boldsymbol{p}$ and constraint system $\mathcal{C}$, we define

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}) \stackrel{\mathrm{def}}{=} \{\, c \in \mathcal{C} \mid \boldsymbol{p} \text{ saturates } c \,\};$$

and, for any constraint c and generator system $\mathcal{G} = (R, P)$, we define

$$\mathrm{sat\_gen}(c, \mathcal{G}) \stackrel{\mathrm{def}}{=} \bigl(\{\, \boldsymbol{r} \in R \mid \boldsymbol{r} \text{ saturates } c \,\}, \{\, \boldsymbol{p} \in P \mid \boldsymbol{p} \text{ saturates } c \,\}\bigr).$$

Definition 9. *Let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$*. A constraint* c *is* ϵ-redundant *in* $\mathcal{C}$ *if* $c \in \mathcal{C}_{>}$ *and at least one of the following conditions holds:*

$$\operatorname{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq (\mathcal{G}_R, \varnothing);$$
$$\exists c' \in \mathcal{C}_{>} \setminus \{c\} \,.\, \operatorname{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq \operatorname{sat\_gen}(c', \mathcal{G}).$$

A generator $\boldsymbol{p}$ *is* ϵ-redundant *in* $\mathcal{G}$ *if* $\boldsymbol{p} \in \mathcal{G}_P$ *and*

$$\exists \boldsymbol{p}' \in \mathcal{G}_P \setminus \{\boldsymbol{p}\} \,.\, \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geq}) \subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}).$$

Note that, according to the above definition, only the strict inequality encodings and the point encodings of an ϵ-polyhedron can be identified as ϵ-redundant constraints and generators, respectively. The following result shows that such a restriction is unconsequential, because all the redundant non-strict inequality encodings and all the redundant closure point encodings will be filtered away by the usual minimization procedure.

Proposition 5. *Let* $\mathcal{R}, \mathcal{R}' \in \mathbb{CP}_{n+1}$ *and* $\mathcal{P} \in \mathbb{P}_n$ *be such that* $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ *and* $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$*. Then*

1. *for any constraint* $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr)$, $\mathcal{R} \subseteq \operatorname{con}\bigl(\{c\}\bigr)$ *if and only if* $\mathcal{R}' \subseteq \operatorname{con}\bigl(\{c\}\bigr)$*;*
2. *for any vector* $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathbb{R}^{n+1}$, $\boldsymbol{p} \in \mathcal{R}$ *if and only if* $\boldsymbol{p} \in \mathcal{R}'$*.*

The next proposition shows that ϵ-redundant constraints and generators can be safely removed from the descriptions of an ϵ-polyhedron without affecting the represented NNC polyhedron. Also, if the ϵ-polyhedron is constraint- or generator-biased, it remains constraint- or generator-biased, respectively.

Proposition 6. *Let* $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$*. Let* $\mathcal{R} \in \mathbb{CP}_{n+1}$ *and* $\mathcal{P} \in \mathbb{P}_n$ *be such that* $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$ *and let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ *be a DD pair in minimal form. Then the following hold:*

1. *If* c *is* ϵ*-redundant in* $\mathcal{C}$*, then* c *is unmatched in* $\mathcal{C}$ *and*
$$\operatorname{con}\bigl(\mathcal{C} \setminus \{c\} \cup \{\epsilon \leq 1\}\bigr) \Rrightarrow_Y \mathcal{P}.$$
2. *If* $\boldsymbol{p}$ *is* ϵ*-redundant in* $\mathcal{G} = (R, P)$*, then* $\boldsymbol{p}$ *is unmatched in* $\mathcal{G}$ *and*
$$\operatorname{gen}\Bigl(\bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)\Bigr) \Rrightarrow_Y \mathcal{P}.$$

If there are no ϵ-redundant constraints or generators, then the constraint or generator system, respectively, is in strong minimal form.

Proposition 7. *Let* $\mathcal{R} \in \mathbb{CP}_{n+1}$ *and* $\mathcal{P} \in \mathbb{P}_n$ *be such that* $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ *and let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ *be a DD pair in minimal form. Then the following hold:*

1. *If* $\mathcal{C}$ *contains no* ϵ*-redundant constraint, then it is in* smf*;*
2. *If* $\mathcal{G}$ *contains no* ϵ*-redundant generator, then it is in* smf*.*

It must be stressed that the above generalizations of the results regarding strong minimal forms to any ϵ-polyhedron are extremely important from the point of view of efficiency. As a matter of fact, a lot of ϵ-redundant constraints and generators may be produced by a few applications of the usual operators, even when starting from descriptions that are in strong minimal form.

To exemplify such a possibility, in Table 2 we report the results obtained for a rather simple experimental evaluation, for which we have adopted the new generator-biased implementation made available by the *Parma Polyhedra Library* [1, 2]. The table has twelve rows in four groups of three. For each triple of rows, we considered four NNC polyhedra $\mathcal{P}_i$ defined by an extended generator system $\mathcal{G}_i = (\varnothing, P_i, C_i)$. All four $\mathcal{G}_i$, which are in minimal form, have the same cardinalities for the P_i and the C_i and these are given in the 1st column. The goal is to compute the NNC polyhedron $\mathcal{P} = (\mathcal{P}_1 \cap \mathcal{P}_2) \uplus (\mathcal{P}_3 \cap \mathcal{P}_4)$ and each row in the triple achieves this by following a different evaluation strategy.

$\# P_i + \# C_i$	eval	Inters ($\# \mathcal{C}_i$)		Poly-hull ($\# \mathcal{G}_{ij}$)		Final result ($\# \mathcal{C}$)			
		1st arg	2nd arg	1st arg	2nd arg	res	smf	time	time-smf
4 + 8	a	48	48	131	77	356	**56**	0.91	0.01
	b	**32**	**32**	**40**	**17**	156	**56**	0.08	0.00
	c	48	**32**	132	**17**	251	**56**	0.16	0.00
8 + 8	a	62	62	209	125	537	**59**	2.29	0.01
	b	**36**	**36**	**50**	**21**	308	**59**	0.25	0.00
	c	62	**36**	190	**21**	332	**59**	0.37	0.00
8 + 10	a	132	132	414	305	2794	**227**	118.64	0.45
	b	**68**	**68**	**58**	**25**	1084	**227**	1.42	0.06
	c	132	**68**	261	**25**	1423	**227**	3.96	0.08
16 + 10	a	178	178	697	657	5078	**235**	932.72	2.07
	b	**80**	**80**	**78**	**29**	1775	**235**	5.24	0.14
	c	178	**80**	418	**29**	1238	**235**	9.48	0.08

Table 2. Exploiting smf's to improve the efficiency of the computation.

For all evaluation strategies, in order to compute the two intersections, we first obtain the constraint systems $\mathcal{C}_i$; the 3rd and 4th columns of the table give the cardinalities of each $\mathcal{C}_i$, where the column labeled '1st arg' indicates $\# \mathcal{C}_1$ and $\# \mathcal{C}_3$ (which are always the same) while that labeled '2nd arg' indicates $\# \mathcal{C}_2$ and $\# \mathcal{C}_4$ (which are also always the same). We then compute the two intersections and obtain the generator systems $\mathcal{G}_{12}$ and $\mathcal{G}_{34}$, whose cardinalities are reported in the 5th and 6th columns. Then, we compute the poly-hull $\mathcal{P}$. In the last four columns we report: the cardinality of the constraint system obtained for $\mathcal{P}$; the same, but after the removal of the ϵ-redundant constraints; the time spent by the overall computation; the time spent to compute the final smf. Rows labeled 'a' correspond to the usual evaluation strategy, where no smf is computed. In this case, the two intersections are computed incrementally: namely, we start from the DD pair of the first argument and incrementally add the constraints of the second argument, keeping the generator system of the result up-to-date so that it will be ready for the following poly-hull computation. Then, we compute the poly-hull, again incrementally. In the rows labeled 'b' we report the outcomes of an evaluation strategy that exploits the possibility of obtaining the smf's of both arguments before each operation (in the table, the cardinalities computed after the application of strong minimization are shown in boldface). Note that, in our current implementation, such a strategy does not fit very well with the adoption of the incremental approach, because after the computation of the smf for one description we no longer have a DD pair [2]. Therefore, after computing the smf of the first argument of each operation, the corresponding dual description has to be recomputed from scratch. The rows labeled 'c' report the outcomes of an intermediate evaluation strategy, where we only compute the smf's of the second argument of each operation, so that the incremental approach can still be adopted. For the examples considered, the latter strategy results in slightly smaller, but still impressive, performance improvements.

Even though the considered examples are not meant to provide a faithful representation of typical computation patterns, we can make a couple of observations based on these experiments. There may be many ϵ-redundant constraints/generators, and their removal can lead to dramatic speed-ups. Moreover, the identification of ϵ-redundant constraints/generators has a negligible cost (see the last column in Table 2) so that the number of ϵ-redundant elements contained in a description can be efficiently computed at run time; based on this, it is always possible to dynamically select the evaluation strategy that is likely to result in a more efficient computation. We believe that the third strategy, by preserving incrementality, is a safe and generally rewarding choice.

5 Conclusion

Convex polyhedra provide the basis for several abstractions used in static analysis and computer-aided verification of complex system. Some of these applications require the manipulation of convex polyhedra that are not necessarily closed. In a previous paper we proposed an elegant way of decoupling the essential geometric features of NNC polyhedra from their traditional implementation. This separation, besides providing a natural and easy to use interface, enables the search for new implementation techniques and makes their eventual integration into existing software libraries seamless (i.e., transparent to the client application). In this work we have shown that the standard implementation of NNC polyhedra, which happens to be biased for constraint-intensive computations, has a dual. We have completely defined this new, previously unknown, implementation and showed that it is biased for generator-intensive computations. Moreover, we have provided a generalization of a notion of strong minimal form that is applicable to both constraint- and generator-biased implementations. We have also shown that this general notion of strong minimization can have a dramatic effect on the size of the representations and, thus, on the efficiency of the algorithms operating upon them.

The encoding based on G-ϵ-polyhedra has dual properties with respect to the one based on C-ϵ-polyhedra. In particular, using a C-ϵ-polyhedron, the encoding of an NNC polyhedron may require a similar number of constraints but as many as twice the number of generators, while, using a G-ϵ-polyhedron, it may require a similar number of generators but as many as twice the number of constraints. We have extended the *Parma Polyhedra Library* [1, 2], a modern C++ library for the manipulation of convex polyhedra, so as to implement NNC polyhedra both with the constraint- and the generator-biased encodings. This enabled us to perform some very preliminary experiments on purely synthetic benchmarks. It seems likely that the performance of one encoding with respect to the other will depend on the particular application and, more specifically, on the kind of polyhedra and operations that are more common in that application.

For future work, given the dual characteristics of the two representations, it would be interesting to investigate whether efficient techniques can be devised so as to use both constraint- and generator-biased encodings, switching dynamically from one to the other in an attempt to maximize performance.

References

1. R. Bagnara, P. M. Hill, E. Ricci, and E. Zaffanella. *The Parma Polyhedra Library User's Manual*. Department of Mathematics, University of Parma, Parma, Italy, release 0.4.2 edition, October 2002. Available at `http://www.cs.unipr.it/ppl/`.
2. R. Bagnara, E. Ricci, E. Zaffanella, and P. M. Hill. Possibly not closed convex polyhedra and the Parma Polyhedra Library. In M. V. Hermenegildo and G. Puebla, editors, *Static Analysis: Proceedings of the 9th International Symposium*, volume 2477 of *Lecture Notes in Computer Science*, pages 213–229, Madrid, Spain, 2002. Springer-Verlag, Berlin.
3. F. Besson, T. P. Jensen, and J.-P. Talpin. Polyhedral analysis for synchronous languages. In A. Cortesi and G. Filé, editors, *Static Analysis: Proceedings of the 6th International Symposium*, volume 1694 of *Lecture Notes in Computer Science*, pages 51–68, Venice, Italy, 1999. Springer-Verlag, Berlin.
4. M. A. Colón and H. B. Sipma. Synthesis of linear ranking functions. In T. Margaria and W. Yi, editors, *Proceedings of the 7th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2001)*, volume 2031 of *Lecture Notes in Computer Science*, pages 67–81, Genova, Italy, 2001. Springer-Verlag, Berlin.
5. P. Cousot and N. Halbwachs. Automatic discovery of linear restraints among variables of a program. In *Conference Record of the Fifth Annual ACM Symposium on Principles of Programming Languages*, pages 84–96, Tucson, Arizona, 1978. ACM Press.
6. N. Halbwachs, A. Kerbrat, and Y.-E. Proy. *POLyhedra INtegrated Environment*. Verimag, France, version 1.0 of POLINE edition, September 1995. Documentation taken from source code.
7. N. Halbwachs, Y.-E. Proy, and P. Raymond. Verification of linear hybrid systems by means of convex approximations. In B. Le Charlier, editor, *Static Analysis:*

Proceedings of the 1st International Symposium, volume 864 of *Lecture Notes in Computer Science*, pages 223–237, Namur, Belgium, 1994. Springer-Verlag, Berlin.
8. N. Halbwachs, Y.-E. Proy, and P. Roumanoff. Verification of real-time systems using linear relation analysis. *Formal Methods in System Design*, 11(2):157–185, 1997.
9. B. Jeannet. *Convex Polyhedra Library*, release 1.1.3c edition, March 2002. Documentation of the "New Polka" library available at `http://www.irisa.fr/prive/Bertrand.Jeannet/newpolka.html`.
10. T. S. Motzkin, H. Raiffa, G. L. Thompson, and R. M. Thrall. The double description method. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games – Volume II*, number 28 in Annals of Mathematics Studies, pages 51–73. Princeton University Press, Princeton, New Jersey, 1953.
11. J. Stoer and C. Witzgall. *Convexity and Optimization in Finite Dimensions I*. Springer-Verlag, Berlin, 1970.

A Proofs

For each set $S \subseteq \mathbb{R}^n$ of finite cardinality m, we denote by $\operatorname{matrix}(S) \subseteq \mathbb{R}^{n \times m}$ the set of all matrices having S as the set of their columns.

In the proofs below we assume the following simple consequence of well known theorems by Minkowski and Weyl [11].

Theorem 1. *The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a closed polyhedron if and only if there exist finite sets $R, P \subseteq \mathbb{R}^n$ of cardinality r and p, respectively, such that $\mathbf{0} \notin R$ and, for any matrices $K \in \mathbb{R}^{n \times r}$ and $L \in \mathbb{R}^{n \times p}$ where $K \in \operatorname{matrix}(R)$ and $L \in \operatorname{matrix}(P)$,*

$$\mathcal{P} = \bigl\{\, K\boldsymbol{\rho} + L\boldsymbol{\pi} \in \mathbb{R}^n \bigm| \boldsymbol{\rho} \in \mathbb{R}^r_+, \boldsymbol{\pi} \in \mathbb{R}^p_+, \textstyle\sum_{i=1}^p \pi_i = 1 \,\bigr\}.$$

Proof (Proof of Proposition 1 on page 6). Let $\mathcal{R} = \operatorname{con}(\mathcal{C})$. We first assume that (4) holds for any constraint $c \in \mathcal{C}$ and show that $\mathcal{R}$ is an ϵ-polyhedron. Condition (2) of Definition 2 holds by hypothesis. We prove condition (3) holds. Let $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$. Then, by (4), $\mathcal{R} \subseteq \operatorname{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr\}\Bigr)$. Thus, for any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ we have $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + 0 \cdot e \geq b$, so that also $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot 0 \geq b$ and hence $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ satisfies c. As c was an arbitrary constraint in $\mathcal{C}$, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and condition (3) holds.

Second we assume that $\mathcal{R}$ is an ϵ-polyhedron and prove that (4) holds. Suppose $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$ and that $\mathcal{R} \subseteq \operatorname{con}\bigl(\{c\}\bigr)$. Then, any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ satisfies c. By condition (3) of Definition 2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and therefore satisfies c. Thus we have $\langle \boldsymbol{a}, \boldsymbol{v} \rangle \geq b$. Hence, if $c_0 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr)$, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ satisfies c_0. As $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ was an arbitrary point in $\mathcal{R}$, $\mathcal{R} \subseteq \operatorname{con}\bigl(\{c_0\}\bigr)$. □

To prove Proposition 2, we need a few additional lemmas.

Lemma 1. *If $\mathcal{R} \in \mathbb{CP}_{n+1}$ is an ϵ-polyhedron and $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}}$ is a ray of $\mathcal{R}$, where $\boldsymbol{r} \neq \mathbf{0}$, then $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$.*

Proof. Since $\mathcal{R}$ has a ray, it is not empty. Thus, let $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. By condition (3) of Definition 2, we also have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$.

Since $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}}$ is a ray of $\mathcal{R}$, for all $\rho \in \mathbb{R}_+$ we have that

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = \bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, \rho e\bigr)^{\mathrm{T}} \in \mathcal{R}.$$

From this, again by condition (3) of Definition 2, we obtain

$$\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, 0\bigr)^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R},$$

proving that also $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$. □

Lemma 2. *If $\mathcal{R} \in \mathbb{CP}_{n+1}$ is an ϵ-polyhedron and $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}}$ is a ray of $\mathcal{R}$, then $e \leq 0$.*

Proof. Since $\mathcal{R}$ is an ϵ-polyhedron, condition (2) of Definition 2 holds so that, for some $\delta > 0$ every point in $\mathcal{R}$ satisfies the constraint $\epsilon \leq \delta$. Since $\mathcal{R}$ has a ray, it is non-empty, so that there exists a point $(\boldsymbol{v}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$ such that $e_0 \leq \delta$. Thus, for all $\rho \in \mathbb{R}_+$,

$$(\boldsymbol{v}_\rho^{\mathrm{T}}, e_\rho)^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}.$$

By condition (2) of Definition 2, $e_\rho = e_0 + \rho e \leq \delta$. Therefore, as this holds for all $\rho \in \mathbb{R}_+$, we have $e \leq 0$. □

Lemma 3. *Let $\mathcal{R} = \operatorname{gen}(\mathcal{G}) \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron, where $\mathcal{G} = (R, P)$. Let also $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$ for some $e' \in \mathbb{R}$ and take $e_{\max} \in \mathbb{R}$ to be the maximal ϵ coordinate such that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$. Then $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \operatorname{gen}\bigl((R', P' \cup C')\bigr)$, where*

$$\begin{aligned}
R' &= \bigl\{\, (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \bigm| e = 0 \,\bigr\}, \\
P' &= \bigl\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e > 0 \,\bigr\}, \\
C' &= \bigl\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e = 0, \forall e' \in \mathbb{R} : (\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \notin P' \,\bigr\}.
\end{aligned}$$

Proof. By hypothesis, $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \operatorname{gen}\bigl((R, P)\bigr)$ so that

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} = (\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}} + \pi_1(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} + \cdots + \pi_p(\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}} \tag{6}$$

where $(\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}}$ is a ray in $\mathcal{R}$, $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\} \subseteq P$, $\pi_1, \ldots, \pi_p > 0$ and $\sum_{i=1}^{p} \pi_i = 1$.

By Lemma 2, $e_r \leq 0$. Note that it cannot be $\boldsymbol{r} = \boldsymbol{0}$, since this would also entail $e_r < 0$, so that

$$(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} = \pi_1(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} + \cdots + \pi_p(\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}} \in \mathcal{R},$$

where $e' = e_{\max} - e_r > e_{\max}$, therefore contradicting the maximality of $e_{\max}$. Hence $\boldsymbol{r} \neq \boldsymbol{0}$ and, by Lemma 1, $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is also a ray in $\mathcal{R}$. Note that it can neither be $e_r < 0$, since in this case, by replacing $(\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}}$ by $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ in (6), we would obtain the same contradiction seen above. Hence $e_r = 0$. Since $(\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}}$

is a ray of $\mathcal{R}$, then it can be obtained as a non-negative combination of rays in R; by Lemma 2, all of these rays must have a non-positive ϵ coordinate and hence, as $e_r = 0$, they all have a zero ϵ coordinate. As a consequence, $(\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}}$ is a non-negative combination of rays in R', so that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \operatorname{gen}\bigl((R', P)\bigr)$.

Suppose that $e_i < 0$ for some $1 \leq i \leq p$. Then, as $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 2, $(\boldsymbol{p}_i^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Replacing the point $(\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}}$ by $(\boldsymbol{p}_i^{\mathrm{T}}, 0)^{\mathrm{T}}$ in (6), we obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} = (\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}} + \pi_1(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} + \cdots + \pi_i(\boldsymbol{p}_i^{\mathrm{T}}, 0)^{\mathrm{T}} + \cdots + \pi_p(\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}} \in \mathcal{R}$$

where $e' = e_{\max} - e_i > e_{\max}$, again contradicting the maximality of $e_{\max}$. Suppose now that, for some $1 \leq i \leq p$, $e_i = 0$ and there exists $e_i' > 0$, such that $(\boldsymbol{p}_i^{\mathrm{T}}, e_i')^{\mathrm{T}} \in P$. Replacing the point $(\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}}$ by $(\boldsymbol{p}_i^{\mathrm{T}}, e_i')^{\mathrm{T}}$ in (6), we obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} = (\boldsymbol{r}^{\mathrm{T}}, e_r)^{\mathrm{T}} + \pi_1(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} + \cdots + \pi_i(\boldsymbol{p}_i^{\mathrm{T}}, e_i')^{\mathrm{T}} + \cdots + \pi_p(\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}} \in \mathcal{R}$$

where $e' = e_{\max} + e_i' > e_{\max}$, which again contradicts the maximality of $e_{\max}$. It follows that, for all $1 \leq i \leq p$, the point $(\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}}$ is in P' if $e_i > 0$ and it is in C' otherwise. Thus, $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \operatorname{gen}\bigl((R', P' \cup C')\bigr)$. $\square$

Proof (Proof of Proposition 2 on page 7). Let $\operatorname{con\_enc}(\mathcal{C})$ be as specified in Definition 3. Let also $\mathcal{G} = (R, P)$, so that $\operatorname{gen\_enc}(\mathcal{G}) = (R', P', C')$, where $R' = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\}$, $P' = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\}$, and $C' = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_c\}$.

Suppose first that $[\![\mathcal{R}]\!] = \varnothing$. Then, by Definition 3, $\operatorname{con}\bigl(\operatorname{con\_enc}(\mathcal{C})\bigr) = \varnothing$. Also, by Definition 1, $\mathcal{R} \subseteq \operatorname{con}\bigl(\{\epsilon \leq 0\}\bigr)$ so that, by Definition 3, $P' = \varnothing$ and $\operatorname{gen}\bigl((R', P', C')\bigr) = \varnothing$. Thus 5) holds.

Suppose now that $[\![\mathcal{R}]\!] \neq \varnothing$. We will first prove that $\operatorname{con}\bigl(\operatorname{con\_enc}(\mathcal{C})\bigr) \subseteq [\![\mathcal{R}]\!]$ and $\operatorname{gen}\bigl(\operatorname{gen\_enc}(\mathcal{G})\bigr) \subseteq [\![\mathcal{R}]\!]$. To this end, we assume that one of the following holds:

$$\boldsymbol{v} \in \operatorname{con}\bigl(\operatorname{con\_enc}(\mathcal{C})\bigr), \tag{7}$$
$$\boldsymbol{v} \in \operatorname{gen}\bigl(\operatorname{gen\_enc}(\mathcal{G})\bigr), \tag{8}$$

and, in each case, we show that there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$.

Suppose that (7) holds. If $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}$, then, by Definition 3, there exists $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \operatorname{con\_enc}(\mathcal{C})$, for some $\bowtie \in \{\geq, >\}$. Thus $\langle \boldsymbol{a}, \boldsymbol{v} \rangle \geq b$ and hence $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ satisfies c for all $e \in \mathbb{R}$. If $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$, for some $s < 0$ and $\boldsymbol{a} \neq \boldsymbol{0}$, then, by Definition 3, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \in \operatorname{con\_enc}(\mathcal{C})$. Thus, as (7) holds, $\langle \boldsymbol{a}, \boldsymbol{v} \rangle > b$. If $\bigl(\langle \boldsymbol{0}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$, for some $s < 0$, then, as $[\![\mathcal{R}]\!]$ is non-empty, by Definition 2, $b < 0$. By condition (2) of Definition 2, the set

$$\Bigl\{ \bigl(\langle \boldsymbol{a}, \boldsymbol{v} \rangle - b\bigr) \in \mathbb{R} \Bigm| \exists s < 0 \,.\, \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C} \Bigr\}$$

is non-empty. Let $e_{\min}$ be its smallest element. It follows that $e_{\min} > 0$ and the point $(\boldsymbol{v}^{\mathrm{T}}, e_{\min})^{\mathrm{T}}$ satisfies every constraint $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$ such that $s \leq 0$. Reasoning towards a contradiction, suppose that $(\boldsymbol{v}^{\mathrm{T}}, e_{\min})^{\mathrm{T}} \notin \mathcal{R}$.

As $[\![\mathcal{R}]\!] \neq \varnothing$, by Definition 1, there exists $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$ such that $e_w > 0$; in particular, $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ satisfies all constraints in $\mathcal{C}$. Thus there exists a point $(\boldsymbol{w}_0^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$ which lies on the line segment joining $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ and $(\boldsymbol{v}^{\mathrm{T}}, e_{\min})^{\mathrm{T}}$ that saturates a constraint $\bigl(\langle \boldsymbol{a}', \boldsymbol{x}\rangle + s' \cdot \epsilon \geq b'\bigr) \in \mathcal{C}$ where $s' > 0$. Thus $\langle \boldsymbol{a}', \boldsymbol{w}_0\rangle + s' \cdot e_0 = b'$. However, as $e_0 > 0$, $\langle \boldsymbol{a}', \boldsymbol{w}_0\rangle < b'$ so that $(\boldsymbol{w}_0^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$, contradicting condition (3) of Definition 2. Thus $(\boldsymbol{v}^{\mathrm{T}}, e_{\min})^{\mathrm{T}} \in \mathcal{R}$.

Suppose next that (8) holds. By Theorem 1

$$\boldsymbol{v} = \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i$$

where $\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p$, $\boldsymbol{\gamma} \in \mathbb{R}_+^c$, $\boldsymbol{\pi} \neq \boldsymbol{0}$ and $\sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1$. By Definition 3, $\bigl\{(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{r}_r^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq R$, $\bigl\{(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}}, \ldots, (\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}}\bigr\} \subseteq P$, for some $e_1, \ldots, e_p > 0$, and $\bigl\{(\boldsymbol{c}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{c}_c^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq P$. Letting

$$(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} = \sum_{i=1}^{r} \rho_i (\boldsymbol{r}_i^{\mathrm{T}}, 0)^{\mathrm{T}} + \sum_{i=1}^{p} \pi_i (\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}} + \sum_{i=1}^{c} \gamma_i (\boldsymbol{c}_i^{\mathrm{T}}, 0)^{\mathrm{T}}$$

we obtain $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathrm{gen}(\mathcal{G}) = \mathcal{R}$. Since $\boldsymbol{\pi} \neq \boldsymbol{0}$, we also obtain $e > 0$.

We now prove that $[\![\mathcal{R}]\!] \subseteq \mathrm{con}\bigl(\mathrm{con\_enc}(\mathcal{C})\bigr)$ and $[\![\mathcal{R}]\!] \subseteq \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G})\bigr)$. To this end, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$, where $e > 0$; since $\mathcal{R}$ is an ϵ-polyhedron, by condition (2) of Definition 2, the ϵ dimension is bounded from above and thus there exists a value $e_{\max} \geq e$ such that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$ and, for all $e' > e_{\max}$, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \notin \mathcal{R}$. We show that both (7) and (8) hold.

Suppose that $c' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\bigr) \in \mathrm{con\_enc}(\mathcal{C})$, where $\bowtie \in \{\geq, >\}$. Then, by Definition 3, there exists $s \leq 0$ such that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$. Since $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$, then $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot e_{\max} \geq b$ so that, as $e_{\max} > 0$ holds, we obtain $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geq b$. Moreover, if $\bowtie \in \{>\}$, then $s < 0$ and we obtain $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$. Thus, for any $\bowtie \in \{\geq, >\}$, $\boldsymbol{v}$ satisfies c'. As $c' \in \mathrm{con\_enc}(\mathcal{C})$ was chosen arbitrarily, (7) holds.

We next prove (8). Since $e_{\max}$ was chosen to be maximal for $\boldsymbol{v}$, we can apply Lemma 3, so that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathrm{gen}\bigl((R'', P'' \cup C'')\bigr)$, where

$$\begin{aligned}
R'' &= \bigl\{\, (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \bigm| e = 0 \,\bigr\} = \bigl\{(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{r}_r^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\}, \\
P'' &= \bigl\{\, (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e > 0 \,\bigr\} = \bigl\{(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}}, \ldots, (\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}}\bigr\}, \\
C'' &= \bigl\{\, (\boldsymbol{c}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \bigm| e = 0, \forall e' > 0 : (\boldsymbol{c}^{\mathrm{T}}, e')^{\mathrm{T}} \notin P'' \,\bigr\} \\
&= \bigl\{(\boldsymbol{c}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{c}_c^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\}.
\end{aligned}$$

By definition of gen, we obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} = \sum_{i=1}^{r} \rho_i (\boldsymbol{r}_i^{\mathrm{T}}, 0)^{\mathrm{T}} + \sum_{i=1}^{p} \pi_i (\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}} + \sum_{i=1}^{c} \gamma_i (\boldsymbol{c}_i^{\mathrm{T}}, 0)^{\mathrm{T}};$$

where $\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p$ and $\boldsymbol{\gamma} \in \mathbb{R}_+^c$ such that $\boldsymbol{\pi} \neq \boldsymbol{0}$ and $\sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1$. By Definition 3, for each $1 \leq i \leq r$, $\boldsymbol{r}_i \in R'$; for $1 \leq i \leq p$, $\boldsymbol{p}_i \in P'$; and, for

$1 \leq i \leq c$, $\boldsymbol{c}_i \in C'$, so that

$$\boldsymbol{v} = \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i$$

and hence $\boldsymbol{v} \in \operatorname{gen}\bigl((R', P', C')\bigr)$. Thus (8) holds. $\square$

Proof (Proof of Proposition 3 on page 10). To prove item 1 of the Proposition, we first show that, for any $Y \in \{C, G\}$,

$$\operatorname{con}(\mathcal{C}) = \operatorname{con}\Bigl(\operatorname{con\_enc}\bigl(\operatorname{con\_repr}_Y(\mathcal{C})\bigr)\Bigr). \tag{9}$$

Let $\mathcal{C}_1 = \operatorname{con\_repr}_Y(\mathcal{C})$ and $\mathcal{C}_2 = \operatorname{con\_enc}(\mathcal{C}_1)$. Let $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathcal{C}$, for some $\bowtie \in \{\geq, >\}$. If $\bowtie \in \{>\}$, then, by Definitions 5 and 7, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b\bigr) \in \mathcal{C}_1$ and hence, by Definition 3, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \in \mathcal{C}_2$. If otherwise $\bowtie \in \{\geq\}$, then, by Definitions 5 and 7, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}_1$ and hence, by Definition 3, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathcal{C}_2$, for some $\bowtie \in \{\geq, >\}$. Thus c is satisfied by all the points in $\operatorname{con}(\mathcal{C}_2)$. As c was an arbitrary constraint in $\mathcal{C}$, we obtain $\operatorname{con}(\mathcal{C}) \subseteq \operatorname{con}(\mathcal{C}_2)$.

Now let $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathcal{C}_2$, for some $\bowtie \in \{\geq, >\}$. If $\bowtie \in \{>\}$, then, by Definition 3, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}_1$, where $s < 0$. By Definitions 5 and 7, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \in \mathcal{C}$. If $\bowtie \in \{\geq\}$, then, by Definition 3, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}_1$. Thus, by Definitions 5 and 7, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathcal{C}$, for some $\bowtie \in \{\geq, >\}$. Thus c is satisfied by all the points in $\operatorname{con}(\mathcal{C})$. As c was an arbitrary constraint in $\mathcal{C}$, we obtain $\operatorname{con}(\mathcal{C}_2) \subseteq \operatorname{con}(\mathcal{C})$. Thus (9) holds. As a consequence, by Proposition 2, we have $\operatorname{con}(\mathcal{C}_1) \Rrightarrow_\epsilon \operatorname{con}(\mathcal{C})$.

Suppose now that $Y = C$. Then, by Definition 5, $\operatorname{con}(\mathcal{C}_1) \subseteq \operatorname{con}\bigl(\{\epsilon \geq 0\}\bigr)$ so that, by Definition 4, $\operatorname{con}(\mathcal{C}_1)$ is a C-ϵ-polyhedron. Otherwise, suppose that $Y = G$. If $\operatorname{con}(\mathcal{C}_1) = \varnothing$ then, by Definition 6, it is a G-ϵ-polyhedron. Otherwise, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \operatorname{con}(\mathcal{C}_1)$ and consider $c' \in \mathcal{C}_1$. By Definition 7, either $c' = (\epsilon \leq 1)$ or, for some $\boldsymbol{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $s \in \{0, -1\}$, $c' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$. Thus, for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon)$ satisfies c', so that $-\boldsymbol{e}_\epsilon$ is a ray in $\operatorname{con}\bigl(\{c'\}\bigr)$. As the choice of c' was arbitrary, $-\boldsymbol{e}_\epsilon$ is a ray in $\operatorname{con}(\mathcal{C}_1)$ so that, by Definition 6, $\operatorname{con}(\mathcal{C}_1)$ is a G-ϵ-polyhedron. Therefore the proof of item 1 is complete.

To prove item 2 of the Proposition, we show that for any $Y \in \{C, G\}$,

$$\operatorname{gen}(\mathcal{G}) = \operatorname{gen}\Bigl(\operatorname{gen\_enc}\bigl(\operatorname{gen\_repr}_Y(\mathcal{G})\bigr)\Bigr). \tag{10}$$

Let $\mathcal{G} = (R, P, C)$, $\mathcal{G}_1 = \operatorname{gen\_repr}_Y(\mathcal{G}) = (R_1, P_1)$ and $\mathcal{G}_2 = \operatorname{gen\_enc}(\mathcal{G}_1) = (R_2, P_2, C_2)$. Suppose first that $\boldsymbol{v} \in R \cup P \cup C$. If $\boldsymbol{v} \in R$, then, by Definitions 5 and 7, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R_1$ and hence, by Definition 3, $\boldsymbol{v} \in R_2$. If $\boldsymbol{v} \in P$, then, by Definitions 5 and 7, $(\boldsymbol{v}^{\mathrm{T}}, 1)^{\mathrm{T}} \in P_1$ and hence, by Definition 3, $\boldsymbol{v} \in P_2$. If $\boldsymbol{v} \in C$, then, by Definitions 5 and 7, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P_1$ and hence, by Definition 3, $\boldsymbol{v} \in P_2 \cup C_2$. Therefore, by definition of gen, we obtain $\operatorname{gen}(\mathcal{G}) \subseteq \operatorname{gen}(\mathcal{G}_2)$.

Now suppose $\boldsymbol{v} \in R_2 \cup P_2 \cup C_2$. If $\boldsymbol{v} \in R_2$, then, by Definition 3, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R_1$. By Definitions 5 and 7, $\boldsymbol{v} \in R$. If $\boldsymbol{v} \in P_2$, then, by Definition 3, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P_1$, for some $e > 0$. Thus, by Definitions 5 and 7, $\boldsymbol{v} \in P$. If $\boldsymbol{v} \in C_2$, then, by Definition 3, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P_1$, for some $e \geq 0$. Thus, by Definitions 5 and 7, $\boldsymbol{v} \in P \cup C$. Therefore, by definition of gen, we obtain $\operatorname{gen}(\mathcal{G}) \supseteq \operatorname{gen}(\mathcal{G}_2)$. Thus (10) holds. As a consequence, by Proposition 2, we have $\operatorname{gen}(\mathcal{G}_1) \Rrightarrow_\epsilon \operatorname{gen}(\mathcal{G})$.

Suppose now that $Y = G$. Then, by Definition 7, $-\boldsymbol{e}_\epsilon \in R_1$ so that, by Definition 6, $\operatorname{gen}(\mathcal{G}_1)$ is a G-ϵ-polyhedron. Otherwise, suppose that $Y = C$. Observe that, by Definition 5, for each vector $(v_1^{\mathrm{T}}, e_1)^{\mathrm{T}} \in R_1 \cup P_1$, we have $e_1 \geq 0$; this implies that, for any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \operatorname{gen}(\mathcal{G}_1)$, we still have $e \geq 0$. Thus $\operatorname{gen}(\mathcal{G}_1) \subseteq \operatorname{con}\bigl(\{\epsilon \geq 0\}\bigr)$ so that, by Definition 4, $\operatorname{gen}(\mathcal{G}_1)$ is a C-ϵ-polyhedron, This completes the proof of item 2. □

The proof of the Proposition 4 on page 10, requires a number of additional preliminary results. In [2], we have shown that when considering NNC polyhedra, closure points can be characterized by a property which is similar to the one used when defining rays.

Proposition 8. [2, Proposition 3] *Let $\mathcal{P} \in \mathbb{P}_n$ and $\boldsymbol{c} \in \mathbb{R}^n$. Then $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$ if and only if $\mathcal{P} \neq \varnothing$ and $\sigma\boldsymbol{p} + (1-\sigma)\boldsymbol{c} \in \mathcal{P}$ for every point $\boldsymbol{p} \in \mathcal{P}$ and $\sigma \in \mathbb{R}$ such that $0 < \sigma < 1$.*

As for a C-ϵ-polyhedron (see [2, Lemma 5]), for any ϵ-polyhedron, closure points in the NNC polyhedron are represented by points lying on the hyperplane defined by $\epsilon = 0$.

Lemma 4. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Then*

$$\mathbb{C}(\mathcal{P}) = \bigl\{\, \boldsymbol{v} \in \mathbb{R}^n \bigm| (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \,\bigr\}.$$

Proof. Letting $\mathcal{P}' = \bigl\{\, \boldsymbol{v} \in \mathbb{R}^n \bigm| (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \,\bigr\}$, we will prove $\mathcal{P}' = \mathbb{C}(\mathcal{P})$.

First, we show that $\mathcal{P}' \subseteq \mathbb{C}(\mathcal{P})$. Let $\boldsymbol{v} \in \mathcal{P}'$, so that $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$, and consider any point $\boldsymbol{p} \in \mathcal{P}$ (note that such a point exists by hypothesis). Then, since $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, there exists $e > 0$ such that $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Since $\mathcal{R}$ is a convex set, for all $\sigma \in \mathbb{R}$ such that $0 < \sigma < 1$ we have

$$\sigma(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} + (1-\sigma)(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} = \bigl(\sigma\boldsymbol{p}^{\mathrm{T}} + (1-\sigma)\boldsymbol{v}^{\mathrm{T}}, \sigma e\bigr)^{\mathrm{T}} \in \mathcal{R}.$$

Since $\sigma e > 0$, by Definition 1, we obtain $\sigma\boldsymbol{p} + (1-\sigma)\boldsymbol{v} \in \mathcal{P}$. As the choices of $\boldsymbol{p} \in \mathcal{P}$ and σ were both arbitrary, we can apply Proposition 8 and conclude $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$.

Now we show that $\mathbb{C}(\mathcal{P}) \subseteq \mathcal{P}'$. Let $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$ and, for all $i \in \mathbb{N}$ such that $i > 1$, define $\sigma_i = \frac{1}{i}$, so that $0 < \sigma_i < 1$. Then, by Proposition 8, for all $\boldsymbol{p} \in \mathcal{P}$ we have

$$\boldsymbol{v}_i = \sigma_i\boldsymbol{p} + (1-\sigma_i)\boldsymbol{v} \in \mathcal{P}.$$

Since $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, by applying the fact that $\mathcal{P} = [\![\mathcal{R}]\!]$ and then property (3) of Definition 2, we obtain $(\boldsymbol{v}_i^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. If $\boldsymbol{p} = \boldsymbol{v}$, then $\boldsymbol{v}_i = \boldsymbol{v}$, so that the thesis holds. Otherwise, let $\boldsymbol{p} \neq \boldsymbol{v}$. For any open ball of $\mathbb{R}^{n+1}$ centered in $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and having radius $\delta > 0$, there exists a $j \in \mathbb{N}$ such that $\sigma_j < \delta$; thus, $(\boldsymbol{v}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ belongs to the ball and, as the choice of δ is arbitrary, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathbb{C}(\mathcal{R})$. However, $\mathcal{R} \in \mathbb{CP}_{n+1}$ is a topologically closed set, so that $\mathcal{R} = \mathbb{C}(\mathcal{R})$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Hence, $\boldsymbol{v} \in \mathcal{P}'$, completing the proof. □

Lemma 5. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and $\boldsymbol{r} \in \mathbb{R}^n$ be a ray of the NNC polyhedron $\mathcal{P}$. Then $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$.*

Proof. Suppose $\boldsymbol{v} \in \mathcal{P}$ and $\rho \in \mathbb{R}_+$. Then $\boldsymbol{v} + \rho\boldsymbol{r} \in \mathcal{P}$. By Definition 1, for some $e_1, e_2 > 0$, we have $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}$ and $\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, e_2\bigr)^{\mathrm{T}} \in \mathcal{R}$ and hence, by condition (2) of Definition 2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and $\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, 0\bigr)^{\mathrm{T}} \in \mathcal{R}$. Thus $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Therefore, as this holds for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$. □

Lemma 6. *Let $\mathcal{R} = \operatorname{gen}\bigl((R, P)\bigr) \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Let also*

$$R' = \bigl\{\, (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R, \boldsymbol{r} \neq \boldsymbol{0} \,\bigr\} \cup \bigl\{\, -\boldsymbol{e}_\epsilon \bigm| (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R, e < 0 \,\bigr\}.$$

Then $\operatorname{gen}\bigl((R', P)\bigr) \Rrightarrow_\epsilon \mathcal{P}$.

Proof. Let $\mathcal{R}' = \operatorname{gen}\bigl((R', P)\bigr)$.

First, suppose that for all $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ we have $e = 0$. Then, the result holds by observing that, in such a case, we obtain $R' = R$ and thus $\mathcal{R}' = \mathcal{R}$.

Second, suppose that there exists $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ such that $e \neq 0$. By Lemma 2, it holds $e < 0$. It follows from the hypothesis that $(-\boldsymbol{e}_\epsilon) \in R'$.

Consider a ray $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$. If $e = 0$, then $\boldsymbol{r} \neq \boldsymbol{0}$ so that, by hypothesis, $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R'$. If $e < 0$ and $\boldsymbol{r} = \boldsymbol{0}$, then we can write $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = -e(-\boldsymbol{e}_\epsilon)$, where $(-\boldsymbol{e}_\epsilon) \in R'$ and $-e > 0$ is a positive factor. Otherwise, if $e < 0$ and $\boldsymbol{r} \neq \boldsymbol{0}$, then, by the hypothesis, we have $\bigl\{(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}, -\boldsymbol{e}_\epsilon\bigr\} \subseteq R'$ and we can write $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} - e(-\boldsymbol{e}_\epsilon)$. Thus, each element of R can be obtained as a positive combination of elements of $\mathcal{R}'$, therefore proving that $\mathcal{R} \subseteq \mathcal{R}'$ and, by monotonicity, $\mathcal{P} \subseteq [\![\mathcal{R}']\!]$.

To prove the other inclusion, let $R'' = R' \setminus \{-\boldsymbol{e}_\epsilon\}$ and $\mathcal{R}'' = \operatorname{gen}\bigl((R'', P)\bigr)$. For each ray $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R''$, by hypothesis, we have $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ so that, by Lemma 1, $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is also a ray of $\mathcal{R}$. Hence, $\mathcal{R}'' \subseteq \mathcal{R}$. By the above observations, we obtain that

$$\forall (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' : \exists (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}, \rho \in \mathbb{R}_+ \,.\, (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon). \tag{11}$$

Let now $\boldsymbol{p} \in [\![\mathcal{R}']\!]$, so that there exists $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$ such that $e > 0$. By applying (11), we obtain that $(\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$, where $e_0 = e + \rho > 0$, proving that $\boldsymbol{p} \in [\![\mathcal{R}]\!] = \mathcal{P}$. As the choice of $\boldsymbol{p}$ was arbitrary, $[\![\mathcal{R}']\!] \subseteq \mathcal{P}$.

To complete the proof, we have to show that $\mathcal{R}'$ is an ϵ-polyhedron. Condition (2) of Definition 2 easily follows from (11), because $\mathcal{R}$ is an ϵ-polyhedron: namely, we can consider the same upper bound constraint $\epsilon \leq \delta$ used for $\mathcal{R}$. To prove condition (3) of Definition 2, let $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$. By (11), we have $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon)$, where $(\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$. As $\mathcal{R}$ is an ϵ-polyhedron, we also have $(\boldsymbol{p}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since we already observed that $\mathcal{R} \subseteq \mathcal{R}'$, this completes the proof. □

The following Lemma is proved in [2].

Lemma 7. *For $j \in \{1,2\}$, let $\mathcal{P}_j = \mathrm{gen}\big((R_j, P_j, C_j)\big) \in \mathbb{P}_n$ be a non-empty NNC polyhedron; let also k_j be the cardinality of R_j and $K_j \in \mathrm{matrix}(R_j)$. Then $\boldsymbol{x} \in \mathcal{P}_1 \uplus \mathcal{P}_2$ if and only if there exist $0 \leq \sigma \leq 1$, $\boldsymbol{x}_1 \in \mathbb{C}(\mathcal{P}_1)$, $\boldsymbol{x}_2 \in \mathbb{C}(\mathcal{P}_2)$, $\boldsymbol{\rho}_1 \in \mathbb{R}_+^{k_1}$ and $\boldsymbol{\rho}_2 \in \mathbb{R}_+^{k_2}$ such that*

$$\boldsymbol{x} = \sigma\boldsymbol{x}_1 + (1-\sigma)\boldsymbol{x}_2 + \boldsymbol{\rho}_1 K_1 + \boldsymbol{\rho}_2 K_2,$$

where $(\boldsymbol{x}_1 \in \mathcal{P}_1 \wedge \sigma > 0) \vee (\boldsymbol{x}_2 \in \mathcal{P}_2 \wedge \sigma < 1)$.

Proof (Proof of Proposition 4 on page 10). The case when $\Rrightarrow_Y \in \{\Rrightarrow_C\}$ is proved in [2, Proposition 3]. We prove here the cases when $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_G\}$.

To prove item (1), we assume that $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2$ and show that $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \cap \mathcal{P}_2$.

We first prove condition (2) of Definition 2. As $\mathcal{R}_1$ and $\mathcal{R}_2$ are ϵ-polyhedra there exists $\delta_1, \delta_2 > 0$ such that $\mathcal{R}_1 \subseteq \mathrm{con}\big(\{\epsilon \leq \delta_1\}\big)$ and $\mathcal{R}_2 \subseteq \mathrm{con}\big(\{\epsilon \leq \delta_2\}\big)$. Letting $\delta = \min\{\delta_1, \delta_2\}$ we have $\mathcal{R}_1 \cap \mathcal{R}_2 \subseteq \mathrm{con}\big(\{\epsilon \leq \delta\}\big)$.

To prove condition (3) of Definition 2, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Then, as $\mathcal{R}_1$ and $\mathcal{R}_2$ are ϵ-polyhedra, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Hence $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$.

Having shown that $\mathcal{R}_1 \cap \mathcal{R}_2$ is an ϵ-polyhedron, we next show that it represents $\mathcal{P}_1 \cap \mathcal{P}_2$. By Definition 1, we have to show that $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$ if and only if there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. First, let $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$. By hypothesis, $\mathcal{R}_1 \Rrightarrow_\epsilon \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$, so that $\mathcal{P}_1 = [\![\mathcal{R}_1]\!]$ and $\mathcal{P}_2 = [\![\mathcal{R}_2]\!]$. Hence, by Definition 1, there exist $e_1, e_2 > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, e_2)^{\mathrm{T}} \in \mathcal{R}_2$. Suppose, without loss of generality, that $e_1 \leq e_2$. By condition (3) of Definition 2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Thus, since $\mathcal{R}_2$ is a convex set, $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_2$. Hence, $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Secondly, suppose that there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Then $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_2$. As $\mathcal{P}_1 = [\![\mathcal{R}_1]\!]$ and $\mathcal{P}_2 = [\![\mathcal{R}_2]\!]$, $\boldsymbol{v} \in \mathcal{P}_1$ and $\boldsymbol{v} \in \mathcal{P}_2$, so that $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$. Thus, $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_1 \cap \mathcal{P}_2$.

To prove that $\mathcal{R}_1 \cap \mathcal{R}_2$ is a G-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are G-ϵ-polyhedra, we consider two subcases. If $\mathcal{R}_1 \cap \mathcal{R}_2 = \varnothing$,then there is nothing to prove. If otherwise $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \varnothing$, then we have to prove that $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}_1 \cap \mathcal{R}_2$. Let $\boldsymbol{v}' \in \mathcal{R}_1 \cap \mathcal{R}_2$ and consider, for any $\rho \in \mathbb{R}_+$, the vector $\boldsymbol{v}'_\rho = \boldsymbol{v}' + \rho(-\boldsymbol{e}_\epsilon)$. As $\boldsymbol{v}' \in \mathcal{R}_1 \cap \mathcal{R}_2$, $\boldsymbol{v}' \in \mathcal{R}_1$ and $\boldsymbol{v}' \in \mathcal{R}_2$. As $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-empty G-ϵ-polyhedra, $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}_1$ and $\mathcal{R}_2$ so that, for any $\rho \in \mathbb{R}_+$, $\boldsymbol{v}'_\rho \in \mathcal{R}_1$ and $\boldsymbol{v}'_\rho \in \mathcal{R}_2$. Hence $\boldsymbol{v}'_\rho \in \mathcal{R}_1 \cap \mathcal{R}_2$.

To prove item (2) of the proposition, we assume that $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_2 \neq \varnothing$ and $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2 \neq \varnothing$ and show that $\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \uplus \mathcal{P}_2$.

For $j \in \{1,2\}$, let $\mathcal{P}_j = \mathrm{gen}\big((R_j, P_j, C_j)\big)$, where R_j has cardinality k_j. By Lemma 7, if $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, then for some $0 \leq \sigma \leq 1$, $(\boldsymbol{v_1}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$, $(\boldsymbol{v_2}^{\mathrm{T}}, e_2)^{\mathrm{T}} \in \mathcal{R}_2$, $e'_1, e'_2 \in \mathbb{R}$, $\boldsymbol{r}_1 = \boldsymbol{\rho}_1 K_1$ and $\boldsymbol{r}_2 = \boldsymbol{\rho}_2 K_2$ where $K_1 \in \mathrm{matrix}(R_1)$, $K_2 \in \mathrm{matrix}(R_2)$, $\boldsymbol{\rho}_1 \in \mathbb{R}_+^{k_1}$ and $\boldsymbol{\rho}_2 \in \mathbb{R}_+^{k_2}$, we have

$$\begin{aligned}
\begin{pmatrix}\boldsymbol{v}\\ e\end{pmatrix} &= \sigma\begin{pmatrix}\boldsymbol{v}_1\\ e_1\end{pmatrix} + (1-\sigma)\begin{pmatrix}\boldsymbol{v}_2\\ e_2\end{pmatrix} + \begin{pmatrix}\boldsymbol{r}_1\\ e'_1\end{pmatrix} + \begin{pmatrix}\boldsymbol{r_2}\\ e'_2\end{pmatrix}\\
&= \begin{pmatrix}\sigma\boldsymbol{v}_1 + (1-\sigma)\boldsymbol{v}_2\\ \sigma e_1 + (1-\sigma)e_2\end{pmatrix} + \begin{pmatrix}\boldsymbol{r_1}\\ e'_1\end{pmatrix} + \begin{pmatrix}\boldsymbol{r_2}\\ e'_2\end{pmatrix}.
\end{aligned} \tag{12}$$

We first prove condition (2) of Definition 2. As $\mathcal{R}_1, \mathcal{R}_2$ are ϵ-polyhedra, there exist $\delta_1, \delta_2 > 0$ such that $\mathcal{R}_1 \subseteq \operatorname{con}(\{\epsilon \leq \delta_1\})$ and $\mathcal{R}_2 \subseteq \operatorname{con}(\{\epsilon \leq \delta_2\})$. Suppose that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$ and rewrite it according to (12). By letting $\delta = \max\{\delta_1, \delta_2\}$, we obtain $e_1 \leq \delta$ and $e_2 \leq \delta$. Also, by Lemma 2, $e'_1, e'_2 \leq 0$. Thus $e = \sigma e_1 + (1-\sigma)e_2 + e'_1 + e'_2$ satisfies $e \leq \delta$ so that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \operatorname{con}(\{\epsilon \leq \delta\})$, as required.

To prove condition (3) of Definition 2, suppose that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, so that we can rewrite it according to (12). As $\mathcal{R}_1$ and $\mathcal{R}_2$ are ϵ-polyhedra, $(\boldsymbol{v}_1^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$ and, by Lemma 1, $(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray in R_1 and $(\boldsymbol{r}_2^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray in R_2, so that

$$\begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix} = \sigma \begin{pmatrix} \boldsymbol{v}_1 \\ 0 \end{pmatrix} + (1-\sigma) \begin{pmatrix} \boldsymbol{v}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{r}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{r}_2 \\ 0 \end{pmatrix} \in \mathcal{R}_1 \uplus \mathcal{R}_2.$$

Having shown that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is an ϵ-polyhedron, we next show that it represents $\mathcal{P}_1 \uplus \mathcal{P}_2$. By Definition 1, we have to prove that $\boldsymbol{v} \in \mathcal{P}_1 \uplus \mathcal{P}_2$ if and only if there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$. First suppose that $\boldsymbol{v} \in \mathcal{P}_1 \uplus \mathcal{P}_2$. Then, by Lemma 7, $\boldsymbol{v} = \sigma \boldsymbol{v}_1 + (1-\sigma)\boldsymbol{v}_2 + \boldsymbol{r}_1 + \boldsymbol{r}_2$, for some $0 \leq \sigma \leq 1$, $\boldsymbol{v}_1 \in \mathbb{C}(\mathcal{P}_1)$ and $\boldsymbol{v}_2 \in \mathbb{C}(\mathcal{P}_2)$, where $\boldsymbol{v}_1 \in \mathcal{P}_1$ and $\sigma > 0$ or $\boldsymbol{v}_2 \in \mathcal{P}_2$ and $\sigma < 1$, and $\boldsymbol{r}_1 = \boldsymbol{\rho}_1 K_1$ and $\boldsymbol{r}_2 = \boldsymbol{\rho}_2 K_2$ where $K_1 \in \operatorname{matrix}(R_1)$ and $K_2 \in \operatorname{matrix}(R_2)$, $\boldsymbol{\rho}_1 \in \mathbb{R}_+^{k_1}$ and $\boldsymbol{\rho}_2 \in \mathbb{R}_+^{k_2}$. Suppose, without loss of generality, that $\boldsymbol{v}_1 \in \mathcal{P}_1$ and $\sigma > 0$. As $\mathcal{R}_1 \Rrightarrow_\epsilon \mathcal{P}_1$, there exists $e_1 > 0$ such that $(\boldsymbol{v}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$. As $\mathcal{R}_2$ is an ϵ-polyhedron, by Lemma 4, from $\boldsymbol{v}_2 \in \mathbb{C}(\mathcal{R}_2)$ we obtain $(\boldsymbol{v}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. By Lemma 5, $(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray in $\mathcal{R}_1$ and $(\boldsymbol{r}_2^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray in $\mathcal{R}_2$. Thus, by letting

$$\begin{pmatrix} \boldsymbol{v} \\ e_1 \end{pmatrix} = \sigma \begin{pmatrix} \boldsymbol{v}_1 \\ e_1 \end{pmatrix} + (1-\sigma) \begin{pmatrix} \boldsymbol{v}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{r}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{r}_2 \\ 0 \end{pmatrix},$$

we obtain $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, where $e_1 > 0$ as required. Secondly, suppose that there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, so that we can rewrite it according to (12). As $e > 0$ and $\sigma \geq 0$, either $e_1 > 0$ and $\sigma > 0$ or $e_2 > 0$ and $\sigma < 1$. Without loss of generality, we assume that $e_1 > 0$ and $\sigma > 0$. As $\mathcal{R}_1 \Rrightarrow_\epsilon \mathcal{P}_1$, we have $\boldsymbol{v}_1 \in \mathcal{P}_1$. As $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$, by Lemma 4, $\boldsymbol{v}_2 \in \mathbb{C}(\mathcal{P}_2)$. Thus, by Lemma 7,

$$\boldsymbol{v} = \sigma \boldsymbol{v}_1 + (1-\sigma)\boldsymbol{v}_2 + \boldsymbol{r}_1 + \boldsymbol{r}_2 \in \mathcal{P}_1 \uplus \mathcal{P}_2.$$

Thus, $\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_1 \uplus \mathcal{P}_2$.

To prove that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is a G-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are G-ϵ-polyhedra, since $\mathcal{R}_1 \uplus \mathcal{R}_2 \neq \varnothing$, we have to show that $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}_1 \uplus \mathcal{R}_2$. To this end, it is sufficient to observe that all the rays of $\mathcal{R}_1$ are also rays of $\mathcal{R}_1 \uplus \mathcal{R}_2$ and $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}_1$, because $\mathcal{R}_1$ is a non-empty G-ϵ-polyhedron.

To prove item (3) of the proposition, we assume that $\mathcal{R} \Rrightarrow_Y \mathcal{P}$ and show that $g(\mathcal{R}) \Rrightarrow_Y f(\mathcal{P})$. Observe that, by definition of g, for any $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ we have

$$g\bigl((\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}\bigr) = \bigl(f(\boldsymbol{v})^{\mathrm{T}}, e\bigr)^{\mathrm{T}}.$$

Thus the coefficient of the ϵ coordinate is not affected at all by the affine transformation, so that conditions (2) and (3) of Definition 2 and $f(\mathcal{P}) = [\![g(\mathcal{R})]\!]$

follow trivially from the hypothesis. To complete the proof, we have to show that if $\mathcal{R}$ is a G-ϵ-polyhedron then $g(\mathcal{R})$ is a G-ϵ-polyhedron. If $\mathcal{R} = \varnothing$, then also $g(\mathcal{R}) = \varnothing$ and there is nothing to prove. If otherwise $\mathcal{R} \neq \varnothing$, then $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}$. We have $g(\mathcal{R}) \neq \varnothing$ and the ray is unaffected by the affine transformation, so that $-\boldsymbol{e}_\epsilon$ is also a ray in $g(\mathcal{R})$. □

Proof (Proof of Proposition 5 on page 12). We first prove item 2, which easily follows from two applications of Lemma 4, since $\mathcal{P} \neq \varnothing$. Namely

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \iff \boldsymbol{v} \in \mathbb{C}(\mathcal{P}) \iff (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'.$$

We now prove the *only if* part of item 1. Thus, let $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr)$ and suppose that $\mathcal{R} \subseteq \operatorname{con}\bigl(\{c\}\bigr)$. Consider a point $\boldsymbol{q} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$; by condition (3) of Definition 2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$ so that, by the previous paragraph, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Thus, the vector $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ satisfies c and, since the coefficient of ϵ in c is 0, $\boldsymbol{q}$ also satisfies c. As the choice of $\boldsymbol{q} \in \mathcal{R}'$ is arbitrary, $\mathcal{R}' \subseteq \operatorname{con}\bigl(\{c\}\bigr)$. The *if* part follows by the same reasoning as above, after swapping $\mathcal{R}$ and $\mathcal{R}'$. □

The proof of Proposition 6 requires some additional notation and a few preliminary lemmas.

Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Then, the set of *$\epsilon$-upper-bounds* of the constraint system $\mathcal{C}$ is defined as

$$\mathcal{C}_\epsilon \stackrel{\text{def}}{=} \Bigl\{ \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C} \Bigm| \boldsymbol{a} = \boldsymbol{0}, s < 0 \Bigr\}.$$

A constraint $c \in \mathcal{C}_\epsilon$ will be usually denoted as $\epsilon \leq \delta$. Note that, since $\mathcal{P} \neq \varnothing$, we have $\delta > 0$.

Lemma 8. *Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be a non-empty ϵ-polyhedron. Then $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}$ if and only if $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon$.*

Proof. Suppose that $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}$. Let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Then, for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon) \in \mathcal{R}$. Thus, if $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$, we have $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot (e - \rho) \geq b$ for all $\rho \in \mathbb{R}_+$. Thus $s \leq 0$ so that $\mathcal{C} = (\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon)$.

Now suppose $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon$. This means that, if $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}$, then $s \leq 0$. As $\mathcal{R}$ is non-empty, there exists a point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Also, for all $\rho \in \mathbb{R}_+$, $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot (e - \rho) \geq b$. As our choice of constraint in $\mathcal{C}$ is arbitrary, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon)$ satisfies all constraints in $\mathcal{C}$ and is therefore in $\mathcal{R}$. Thus $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}$. □

Lemma 9. *Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Let also $\mathcal{C}' = \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon \cup \{\epsilon \geq 0\}$ and $\mathcal{C}'' = \mathcal{C} \cup \{\epsilon \geq 0\}$. Then $\operatorname{con}(\mathcal{C}') \Rrightarrow_\epsilon \mathcal{P}$ and $\operatorname{con}(\mathcal{C}') = \operatorname{con}(\mathcal{C}'')$.*

Proof. Let $\mathcal{R}' = \operatorname{con}(\mathcal{C}')$, $\mathcal{R}'' = \operatorname{con}(\mathcal{C}'')$, and $\mathcal{C}^* = \mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon)$. Note that, by Definition 1, we have $\mathcal{P} = [\![\mathcal{R}]\!] = [\![\mathcal{R}'']\!]$. Moreover, by Proposition 1, since $\mathcal{R}$

is an ϵ-polyhedron, $\mathcal{R}''$ is also an ϵ-polyhedron. To complete the proof, we show that $\mathcal{R}' = \mathcal{R}''$.

Observe that $\mathcal{R}'' \subseteq \mathcal{R}'$, because $\mathcal{C}' \subseteq \mathcal{C}''$. We now show the other inclusion $\mathcal{R}' \subseteq \mathcal{R}''$. Let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$, so that $e \geq 0$. By hypothesis, $\mathcal{P} \neq \varnothing$ so that there exists a point $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$ such that $e_w > 0$. By hypothesis, both $\boldsymbol{p}$ and $\boldsymbol{q}$ must satisfy every constraint in $\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon = \mathcal{C}'_> \cup \mathcal{C}'_\geq \cup \mathcal{C}'_\epsilon$. We show that $\boldsymbol{p}$ also satisfies all the constraints in $\mathcal{C}^*$, so that $\boldsymbol{p} \in \mathcal{R}''$, completing the proof. Suppose, by contraposition, that $\boldsymbol{p}$ does not satisfy a constraint in $\mathcal{C}^*$. Let

$$\{\, \sigma \boldsymbol{p} + (1-\sigma)\boldsymbol{q} \mid 0 \leq \sigma \leq 1 \,\}$$

be the set of points lying on the segment between $\boldsymbol{p}$ and $\boldsymbol{q}$. As $\boldsymbol{p} \notin \mathcal{R}''$ and $\boldsymbol{q} \in \mathcal{R}''$, there must exists a minimum value $0 \leq \tau < 1$ such that

$$\boldsymbol{p}' = (\boldsymbol{v}_\tau, e_\tau) = \tau \boldsymbol{p} + (1-\tau)\boldsymbol{q} \in \mathcal{R}'',$$

so that $\boldsymbol{p}'$ saturates some constraint $c^* = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C}^*$. Note that $e_\tau > 0$ and, by definition of $\mathcal{C}^*$, we have $s > 0$. As a consequence, $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ does not satisfy c^*, which implies $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}''$. However, since $\mathcal{R}''$ is an ϵ-polyhedron, this contradics condition (3) of Definition 2. □

Lemma 10. *Let $\mathcal{R} = \mathrm{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let $\boldsymbol{p} \in \mathcal{R}$ be such that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$, where $e > 0$, and consider $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$. Then*

$$\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon) = \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq).$$

Proof. Consider $c \in \mathcal{C}_\geq$, so that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr)$; then $c \in \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq)$ if and only if $c \in \mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq)$, so that $\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) = \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq)$. Consider now $c \in \mathcal{C}_>$, so that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$ where $s < 0$; since $e > 0$, we obtain $\langle \boldsymbol{a}, \boldsymbol{v} \rangle > b$, so that $\boldsymbol{p}_0$ satisfies but does not saturate c; thus $\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_>) = \varnothing$. Consider now $c \in \mathcal{C}_\epsilon$, so that $c = (\epsilon \leq \delta)$ for some $\delta > 0$; then it follows that $\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\epsilon) = \varnothing$. By all the above relations, we obtain

$$\begin{aligned}
&\mathrm{sat_con}(\boldsymbol{p}_0, \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon) \\
&\quad = \mathrm{sat_con}(\boldsymbol{p}_0, \mathcal{C}_>) \cup \mathrm{sat_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) \cup \mathrm{sat_con}(\boldsymbol{p}_0, \mathcal{C}_\epsilon) \\
&\quad = \mathrm{sat_con}(\boldsymbol{p}, \mathcal{C}_\geq).
\end{aligned}$$

□

Lemma 11. *Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let also $c \in \mathcal{C}_>$ be such that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$ and consider $c_0 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr)$. Then*

$$\mathrm{sat\_gen}\bigl(c_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr) = \mathrm{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr).$$

Proof. If $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{G}_C \cup \mathcal{G}_R$, then $e = 0$. Thus, $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot e = b$ if and only if $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + 0 \cdot e = b$. Similarly, $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot e = 0$ if and only if $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + 0 \cdot e = 0$. Thus, if $\boldsymbol{p}$ is a point or $\boldsymbol{p}$ is a ray, $\boldsymbol{p}$ saturates c if and only if it saturates c_0. As $\boldsymbol{p}$ is an arbitrary point or ray in $\mathcal{G}_C \cup \mathcal{G}_R$, we have the required result. □

Lemma 12. *Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let also $c \in \mathcal{C}_>$ be saturated by the point $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \operatorname{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$.*

Proof. Let $\mathcal{G} = (R, P)$ and $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr)$. Then, as $c \in \mathcal{C}_>$, $s < 0$. Since $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ saturates c, it holds $\langle \boldsymbol{a}, \boldsymbol{v}\rangle = b$, so that for all $e > 0$ we have $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin \mathcal{R}$. Therefore we can apply Lemma 3, taking $e_{\max} = 0$, so that we obtain $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \operatorname{gen}\bigl((\mathcal{G}_R, \mathcal{G}_P \cup \mathcal{G}_C)\bigr)$. By definition of gen, we conclude $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \operatorname{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$. □

Lemma 13. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron and $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then, if c is ϵ-redundant in $\mathcal{C}$, c is unmatched in $\mathcal{C}$.*

Proof. Let $\mathcal{G} = (R, P)$. Suppose that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr)$ is ϵ-redundant in $\mathcal{C}$. Let also $c_0 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b\bigr)$ be the constraint matching c. We suppose that $c_0 \in \mathcal{C}$ and derive a contradiction. As c is ϵ-redundant, by Definition 9, $c \in \mathcal{C}_>$ and there are two cases to consider.

First suppose that

$$\text{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq (\mathcal{G}_R, \varnothing). \tag{13}$$

Then, for all $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_C$, $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot 0 > b$. Thus, by Lemma 12, for all $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$, $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot 0 > b$. However, by condition (3) of Definition 2, for all $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$, the point $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Thus, for all $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$, we must have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + 0 \cdot e > b$ so that c_0 is not saturated by any points in $\mathcal{R}$. As $\mathcal{C}$ is minimal, $c_0 \notin \mathcal{C}$ which is a contradiction.

Secondly, we assume that (13) does not hold so that, by Definition 9, there exists $c' = \bigl(\langle \boldsymbol{a}', \boldsymbol{x}\rangle + s' \cdot \epsilon \geq b'\bigr) \in \mathcal{C}_> \setminus \{c\}$ such that

$$\text{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq \text{sat\_gen}(c', \mathcal{G}). \tag{14}$$

Let $c'_0 = \bigl(\langle \boldsymbol{a}', \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b'\bigr)$. As c and c' are in $\mathcal{C}_>$, we have both $s < 0$ and $s' < 0$. By Lemma 11, we have

$$\begin{aligned}
\text{sat_gen}\bigl(c_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr) &= \text{sat_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr), \\
\text{sat_gen}\bigl(c'_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr) &= \text{sat_gen}\bigl(c', (\mathcal{G}_R, \mathcal{G}_C)\bigr)
\end{aligned}$$

so that, by (14),

$$\text{sat\_gen}\bigl(c_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq \text{sat\_gen}\bigl(c'_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr). \tag{15}$$

We show that

$$\text{sat\_gen}(c_0, \mathcal{G}) \sqsubseteq \text{sat\_gen}(c'_0, \mathcal{G}). \tag{16}$$

Let $(\mathcal{H}_R, \mathcal{H}_P) = \text{sat\_gen}\bigl(c_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr)$ and $(\mathcal{H}'_R, \mathcal{H}'_P) = \text{sat\_gen}\bigl(c'_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr)$. Suppose that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \cup R$ saturates c_0 so that $\langle \boldsymbol{a}, \boldsymbol{v}\rangle = b$. Then, to prove (16), we show that $\boldsymbol{p}$ is a point or ray in $\text{sat\_gen}(c'_0, \mathcal{G})$. Let $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ so that $\boldsymbol{p}_0$ saturates c_0 and hence, c. Suppose first that $\boldsymbol{p} \in P$. By condition (3) of Definition 2, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. By Lemma 12, $\boldsymbol{p}_0 \in \operatorname{gen}\bigl((\mathcal{H}_R, \mathcal{H}_C)\bigr)$. Therefore, by (15), $\boldsymbol{p}_0 \in \operatorname{gen}\bigl((\mathcal{H}'_R, \mathcal{H}'_C)\bigr)$ so that $\boldsymbol{p}_0$ saturates c'_0. Thus $\langle \boldsymbol{a}', \boldsymbol{v} \rangle = b'$ and $\boldsymbol{p}$ saturates c'_0. Thus $\boldsymbol{p}$ is a point in $\operatorname{sat\_gen}(c'_0, \mathcal{G})$. Suppose next that that $\boldsymbol{p} \in R$. By Lemma 1, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray in $\mathcal{R}$. By Lemma 2, every ray in R has a non-positive value for the ϵ coordinate so that as $\boldsymbol{p}_0$ is a linear combination of rays in R that saturate c_0, $\boldsymbol{p}_0$ must be a linear combination of rays in $\mathcal{G}_R$ that saturate c_0. Thus $\boldsymbol{p}_0$ is a linear combination of rays in $\mathcal{H}_R$. Therefore, by (15), $\boldsymbol{p}_0$ must be a linear combination of rays in $\mathcal{H}'_R$ so that $\boldsymbol{p}_0$ saturates c'_0. Thus $\langle \boldsymbol{a}', \boldsymbol{v} \rangle = b'$ and $\boldsymbol{p}$ saturates c'_0. Thus $\boldsymbol{p}$ is a ray in $\operatorname{sat\_gen}(c'_0, \mathcal{G})$.

If c'_0 and c_0 are distinct constraints, then c'_0 is a linear combination of constraints including $c'' \in \mathcal{C} \setminus \{c_0\}$ such that $\operatorname{sat\_gen}(c'_0, \mathcal{G}) \sqsubseteq \operatorname{sat\_gen}(c'', \mathcal{G})$. By (16), $\operatorname{sat\_gen}(c_0, \mathcal{G}) \sqsubseteq \operatorname{sat\_gen}(c'', \mathcal{G})$ contradicting the hypothesis that $\mathcal{C}$ is minimal. On the other hand, if c'_0 and c_0 are the same, then $c' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s' \cdot \epsilon \geq b\bigr)$. As $c \neq c'$, either $s > s'$ or $s < s'$. If $s > s'$, every point that satisfies c' also satisfies c so that c is redundant and similarly, if $s < s'$, c' is redundant; both cases contradicting the hypothesis that $\mathcal{C}$ is minimal. ◻

Lemma 14. *Let* $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. *Let* $\mathcal{R} \in \mathbb{CP}_{n+1}$ *be such that* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ *is a DD pair in minimal form and* $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$. *If* c *is an* ϵ*-redundant constraint in* $\mathcal{C}$, *then* c *is unmatched in* $\mathcal{C}$ *and* $\operatorname{con}(\mathcal{C}') \Rrightarrow_Y \mathcal{P}$, *where* $\mathcal{C}' = \mathcal{C} \setminus \{c\} \cup \{\epsilon \leq 1\}$.

Proof. By Lemma 13, c is unmatched in $\mathcal{C}$. By Definition 9, we have $c \in \mathcal{C}_>$; thus $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr)$, where $\boldsymbol{a} \neq \boldsymbol{0}$ and $s < 0$.

Let $\mathcal{R}' = \operatorname{con}(\mathcal{C}')$ and consider $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$. We first show that

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}, \tag{17}$$

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \text{ does not saturate } c. \tag{18}$$

As $\mathcal{P} \neq \varnothing$, by Definition 1, there exists $(\boldsymbol{w}^{\mathrm{T}}, e'_w)^{\mathrm{T}} \in \mathcal{R}$ for some $e'_w > 0$. As $\mathcal{R}$ is an ϵ-polyhedron, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Thus, as $\mathcal{R}$ is a convex set, for some $0 < e_w \leq 1$, $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$. Since $e_w \leq 1$, we also have that $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}'$. Consider the segment between $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$ and $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}' \cap \mathcal{R}$. As $\mathcal{R}'$ is a convex set, for each $0 \leq \sigma \leq 1$, we have $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \in \mathcal{R}'$, where

$$(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} = \sigma(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + (1 - \sigma)(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}. \tag{19}$$

Letting $\sigma = 0$, we obtain $\boldsymbol{v}_0 = \boldsymbol{w}$ so that, by condition (3) of Definition 2, $(\boldsymbol{v}_0^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Now let τ be the maximum value between 0 and 1 such that $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then, for all $\sigma \in \mathbb{R}$ such that $\tau < \sigma \leq 1$, $(\boldsymbol{v}_\sigma^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$. Thus, again by condition (3) of Definition 2, $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e')^{\mathrm{T}} \notin \mathcal{R}$ for all $e' \in \mathbb{R}$ and all $\sigma \in \mathbb{R}$ such that $\tau < \sigma \leq 1$.

Suppose first that $\tau < 1$. Then, it follows that the only point in $\mathcal{R}$ on the line joining $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ and $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ is the end point $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$. As every point on this line is in $\mathcal{R}'$, this implies that $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ saturates constraint c. Thus, by Lemma 12, $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \in \operatorname{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$ so that $\operatorname{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \not\sqsubseteq (\mathcal{G}_R, \varnothing)$. As a consequence, since by hypothesis c is ϵ-redundant in $\mathcal{C}$, by Definition 9 there exists a constraint $c' = \bigl(\langle \boldsymbol{a}', \boldsymbol{x} \rangle + s' \cdot \epsilon \geq b'\bigr) \in \mathcal{C}'_{>}$ such that

$$\text{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq \text{sat\_gen}(c', \mathcal{G}). \tag{20}$$

By (20), the point $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ also saturates c'. As already observed before, for all $\sigma \in \mathbb{R}$ such that $\tau < \sigma \leq 1$, we have $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$, so that $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}}$ does not satisfy c but does satisfy c'. As $s < 0$, $e_\tau \geq 0$. Hence, as $s' < 0$, the point $(\boldsymbol{v}_\tau^{\mathrm{T}}, e_\tau)^{\mathrm{T}}$ either saturates c' or it does not satisfy c'; in both cases, for all $\sigma \in \mathbb{R}$ such that $\tau < \sigma \leq 1$, the point $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \in \mathcal{R}'$ does not satisfy $c' \in \mathcal{C}'$, which is a contradiction. Thus it must hold $\tau = 1$ and, by (19),

$$(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}' \cap \mathcal{R}$$

so that (17) holds. Suppose now that (18) does not hold, so that $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ saturates c. As $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$, it does not satisfy c. Thus, since $s < 0$, we must have $e > 0$. By (20), $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ also saturates c'. Hence, as $s' < 0$, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ does not satisfy c', contradicting the assumption that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$. Therefore (18) holds.

To prove $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$, we show that $\mathcal{R}'$ is an ϵ-polyhedron and $[\![\mathcal{R}]\!] = [\![\mathcal{R}']\!]$.

By taking $\delta = 1$, the inclusion $\mathcal{R}' \subseteq \operatorname{con}\bigl(\{\epsilon \leq \delta\}\bigr)$ holds trivially, because the constraint $\epsilon \leq 1$ has been explicitly added in $\mathcal{C}'$. Thus condition (2) of Definition 2 holds. By (17), if $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ is an arbitrary point in $\mathcal{R}'$, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ obviously satisfies the constraint $\epsilon \leq 1$, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$, so that condition (3) of Definition 2 also holds and $\mathcal{R}'$ is an ϵ-polyhedron.

To prove the inclusion $[\![\mathcal{R}]\!] \subseteq [\![\mathcal{R}']\!]$, let $\boldsymbol{v} \in [\![\mathcal{R}]\!]$. Thus, there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. By condition (3) of Definition 2, we also have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ so that, as $\mathcal{R}$ is a convex set, there exists $0 < e' \leq 1$ such that $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. Note that $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}}$ satisfies all the constraints in $\mathcal{C}$ and it also satisfies the constraint $\epsilon \leq 1$; as a consequence, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}'$ and $\boldsymbol{v} \in [\![\mathcal{R}']\!]$, as required.

To show the other inclusion $[\![\mathcal{R}']\!] \subseteq [\![\mathcal{R}]\!]$, let $\boldsymbol{v} \in [\![\mathcal{R}']\!]$. Thus, there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$. By (17) and (18), $\langle \boldsymbol{a}, \boldsymbol{v} \rangle > b$. Thus, by letting $e' = \min\Bigl(\bigl\{e, \frac{b - \langle \boldsymbol{a}, \boldsymbol{v} \rangle}{s}\bigr\}\Bigr)$, we obtain $e' > 0$ and $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. Thus $\boldsymbol{v} \in [\![\mathcal{R}]\!]$.

Suppose next that $\mathcal{R} \Rrightarrow_C \mathcal{P}$ so that $\mathcal{R}$ is a C-ϵ-polyhedron. By the first part of the proof, $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. We show that $\operatorname{con}(\mathcal{C}') \subseteq \operatorname{con}\bigl(\{\epsilon \geq 0\}\bigr)$. By contraposition, suppose that there exists $(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} \in \mathcal{R}'$ where $e_u < 0$. As $[\![\mathcal{R}']\!] \neq \varnothing$, there exists $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}'$ where $e_w > 0$. By condition (2) of Definition 2, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$. Let $\sigma = \frac{e_w}{2(e_w - e_u)}$ so that, as $e_u < 0$ and $e_w > 0$, we have $0 < \sigma < 1$. Then, if

$$\begin{aligned}
(\boldsymbol{w}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} &= \sigma(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} + (1 - \sigma)(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}, \\
(\boldsymbol{w}_\sigma^{\mathrm{T}}, e'_\sigma)^{\mathrm{T}} &= \sigma(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} + (1 - \sigma)(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}},
\end{aligned}$$

we obtain $e_\sigma > 0$ and $e'_\sigma < 0$ and both $(\boldsymbol{w}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \in \mathcal{R}'$ and $(\boldsymbol{w}_\sigma^{\mathrm{T}}, e'_\sigma)^{\mathrm{T}} \in \mathcal{R}'$. As $e_\sigma > 0$, $\boldsymbol{w}_\sigma \in [\![\mathcal{R}']\!] = [\![\mathcal{R}]\!]$. Thus, by Definition 1 and condition (2) of Definition 2, $(\boldsymbol{w}_\sigma^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and hence satisfies c. Thus, as $s \cdot e'_\sigma > 0$, $(\boldsymbol{w}_\sigma^{\mathrm{T}}, e'_\sigma)^{\mathrm{T}}$

also satisfies c and, as it is in $\mathcal{R}' = \mathrm{con}(\mathcal{C}')$, it satisfies all the constraints in $\mathcal{C}$. Thus $(\boldsymbol{w}_\sigma^{\mathrm{T}}, e'_\sigma)^{\mathrm{T}} \in \mathcal{R}$. However, as $\mathcal{R}$ is a C-ϵ-polyhedron, by Definition 4, $\mathcal{R} \subseteq \mathrm{con}\bigl(\{\epsilon \geq 0\}\bigr)$ which contradicts $e'_\sigma < 0$. Thus $\mathrm{con}(\mathcal{C}') \subseteq \mathrm{con}\bigl(\{\epsilon \geq 0\}\bigr)$ and $\mathcal{R}' \Rrightarrow_C \mathcal{P}$.

Finally, suppose that $\mathcal{R} \Rrightarrow_G \mathcal{P}$. By the first part of the proof $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. Note that, since $\mathcal{P} \neq \varnothing$, we also have $\mathcal{R} \neq \varnothing$ and $\mathcal{R}' \neq \varnothing$. Thus, by Definition 6, $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}$ and we need to show that it is also a ray of $\mathcal{R}'$. By Lemma 8, $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon$. As $c \in \mathcal{C}_>$ and $(\epsilon \leq 1) \in \mathcal{C}'_\epsilon$, we also have $\mathcal{C}' = \mathcal{C}'_> \cup \mathcal{C}'_\geq \cup \mathcal{C}'_\epsilon$ so that, again by Lemma 8, $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}'$. Thus, $\mathcal{R}' \Rrightarrow_G \mathcal{P}$. □

Lemma 15. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron and $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then, if $\boldsymbol{p}$ is ϵ-redundant in $\mathcal{G}$, $\boldsymbol{p}$ is unmatched in $\mathcal{G}$.*

Proof. Let $\mathcal{G} = (R, P)$ and suppose that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ is ϵ-redundant in $\mathcal{G}$. Then, by Definition 9, $\boldsymbol{p} \in \mathcal{G}_P$ and there exists $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{G}_P \setminus \{\boldsymbol{p}\}$ such that

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}). \tag{21}$$

As $\boldsymbol{p}$ and $\boldsymbol{p}'$ are in $\mathcal{G}_P$, we have both $e > 0$ and $e' > 0$. Let $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and $\boldsymbol{p}'_0 = (\boldsymbol{y}^{\mathrm{T}}, 0)^{\mathrm{T}}$; as $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 2, we have both $\boldsymbol{p}_0 \in \mathcal{R}$ and $\boldsymbol{p}'_0 \in \mathcal{R}$. Also note that, since $\mathcal{G}$ is in minimal form, then $\boldsymbol{v} \neq \boldsymbol{y}$.

To prove the result, we assume that $\boldsymbol{p}$ is matched (i.e., $\boldsymbol{p}_0 \in P$) and derive a contradiction. By Lemma 10, we have

$$\begin{aligned}
\mathrm{sat_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) &= \mathrm{sat_con}(\boldsymbol{p}, \mathcal{C}_\geq), \\
\mathrm{sat_con}(\boldsymbol{p}'_0, \mathcal{C}_\geq) &= \mathrm{sat_con}(\boldsymbol{p}', \mathcal{C}_\geq)
\end{aligned}$$

so that, by (21),

$$\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}_\geq). \tag{22}$$

Suppose first that $e \leq e'$ and let $\sigma = \frac{e}{e'}$, so that $0 < \sigma \leq 1$. Consider the point $\boldsymbol{q}_e = \sigma \boldsymbol{p}' + (1 - \sigma)\boldsymbol{p}_0 = (\boldsymbol{w}^{\mathrm{T}}, e)^{\mathrm{T}}$. Being a convex combination of $\boldsymbol{p}'$ and $\boldsymbol{p}_0$, we have $\boldsymbol{q}_e \in \mathcal{R}$. Let also $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q}_e$; then $\boldsymbol{r}$ cannot be a ray of $\mathcal{R}$, since otherwise we would have $\boldsymbol{p} \in \mathrm{gen}\Bigl(\bigl(R, \{\boldsymbol{p}', \boldsymbol{p}_0\}\bigr)\Bigr)$, contradicting the hypothesis that $\mathcal{G}$ is in minimal form. For each $\rho \in \mathbb{R}_+$, let $\boldsymbol{p}_\rho = \boldsymbol{p}_0 + \rho \boldsymbol{r} = (\boldsymbol{s}^{\mathrm{T}}, 0)^{\mathrm{T}}$. Since $\boldsymbol{r}$ is not a ray of $\mathcal{R}$, there must exist $\rho' \in \mathbb{R}_+$ such that $\boldsymbol{p}_{\rho'} \in \mathcal{R}$ but $\boldsymbol{p}_\rho \notin \mathcal{R}$, for all $\rho > \rho'$. If $\rho' > 0$, then $\boldsymbol{p}_{\rho'} \neq \boldsymbol{p}_0$; thus, $\boldsymbol{p}_0$ can be expressed as a convex combination of $\boldsymbol{p}_{\rho'}$ and $\boldsymbol{p}'_0$, contradicting the hypothesis that $\mathcal{G}$ is in minimal form. Therefore, it must hold $\rho' = 0$ (i.e., $\boldsymbol{p}_{\rho'} = \boldsymbol{p}_0$). Since for all $\rho > 0$ we have $\boldsymbol{p}_\rho = (\boldsymbol{s}^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$ and $\mathcal{R}$ is an ϵ-polyhedron, then $(\boldsymbol{s}^{\mathrm{T}}, e'')^{\mathrm{T}} \notin \mathcal{R}$ for all $e'' \in \mathbb{R}$. As a consequence, there must exist a constraint $c \in \mathcal{C}$ such that c is saturated by all the points lying on the segment identified by $\boldsymbol{p}$ and $\boldsymbol{p}_0$. As a consequence, $c \in \mathcal{C}_\geq$ and c is not saturated by $\boldsymbol{p}'_0$. However, this contradicts the condition (22) established above, so that we cannot have $e \leq e'$.

Secondly suppose that $e > e'$ and let $\boldsymbol{p}_{e'} = (\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}}$. Being a convex combination of $\boldsymbol{p}$ and $\boldsymbol{p}_0$, we have $\boldsymbol{p}_{e'} \in \mathcal{R}$. Let also $\boldsymbol{r}' = \boldsymbol{p}' - \boldsymbol{p}_{e'}$; then $\boldsymbol{r}'$ cannot be a ray of $\mathcal{R}$, since otherwise we would have $\boldsymbol{p}' \in \operatorname{gen}\Big(\big(R, \{\boldsymbol{p}, \boldsymbol{p}_0\}\big)\Big)$, contradicting the hypothesis that $\mathcal{G}$ is in minimal form. Consider now the vector $\boldsymbol{r} = -\boldsymbol{r}'$. Again, $\boldsymbol{r}$ cannot be a ray of $\mathcal{R}$, since otherwise $\boldsymbol{p}_0$ could be obtained by combining $\boldsymbol{p}_0'$ and $\boldsymbol{r}$. Thus, we are in the same situation identified before and a contradiction can be derived by the same argument. As a consequence, $\boldsymbol{p}_0 \in P$ cannot hold, so that $\boldsymbol{p}$ is unmatched in $\mathcal{G}$, completing the proof. ☐

Lemma 16. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$, $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ and $\{\boldsymbol{p}, \boldsymbol{p}'\} \subseteq \mathcal{G}_P$, where* $\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}_\geq)$. *Let also $\mathcal{G} = (R, P)$, $\mathcal{G}' = \big(R, P \setminus \{\boldsymbol{p}\}\big)$ and $\mathcal{R}' = \operatorname{gen}(\mathcal{G}')$. Then*

$$\mathcal{R} \cap \operatorname{con}\big(\{\epsilon = 0\}\big) = \mathcal{R}' \cap \operatorname{con}\big(\{\epsilon = 0\}\big).$$

Proof. Since $\mathcal{G}' \sqsubset \mathcal{G}$, we have $\mathcal{R}' \subseteq \mathcal{R}$. Therefore, to prove the lemma, we need to show that any point $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ is also in $\mathcal{R}'$. Note that any ray in $\mathcal{R}$ is also a ray in $\mathcal{R}'$.

Let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$ and $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e_y)^{\mathrm{T}}$ so that, since they are both in $\mathcal{G}_P$, we obtain $e_v > 0$ and $e_y > 0$. Consider $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and $\boldsymbol{p}_0' = (\boldsymbol{y}^{\mathrm{T}}, 0)^{\mathrm{T}}$. As $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 2, we have $\{\boldsymbol{p}_0, \boldsymbol{p}_0'\} \subseteq \mathcal{R}$. Thus, $\boldsymbol{p}_0$ can be rewritten as $\boldsymbol{p}_0 = \sigma\boldsymbol{p} + (1-\sigma)\boldsymbol{p}_-$, where $0 \leq \sigma \leq 1$ and the point $\boldsymbol{p}_- = (\boldsymbol{v}^{\mathrm{T}}, e_-)^{\mathrm{T}}$ is such that $\boldsymbol{p}_- \in \operatorname{gen}(\mathcal{G}') = \mathcal{R}'$. Since $e_v > 0$, we obtain $e_- \leq 0$. Since $\boldsymbol{p}' \in \mathcal{R}'$, which is a convex set, then $\mathcal{R}'$ contains the whole segment $[\boldsymbol{p}_-, \boldsymbol{p}']$ and, in particular, by taking $\boldsymbol{q}_1 = (\boldsymbol{w}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$ to be the point on this segment having a zero ϵ coordinate, we obtain $\boldsymbol{q}_1 \in \mathcal{R}'$ (note that there exists exactly one such a $\boldsymbol{q}_1$, because $e_y > 0$). Thus, by applying Lemma 10 to $\boldsymbol{p}$ and $\boldsymbol{p}'$ we obtain

$$\begin{aligned} \mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) &= \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq), \\ \mathrm{sat\_con}(\boldsymbol{p}_0', \mathcal{C}_\geq) &= \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}_\geq) \end{aligned} \tag{23}$$

so that, by hypothesis,

$$\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}_0', \mathcal{C}_\geq).$$

Thus, as $\boldsymbol{q}_1$ lies on the segment $[\boldsymbol{p}_0, \boldsymbol{p}_0']$, we obtain

$$\mathrm{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_\geq)$$

and hence, using again (23),

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_\geq). \tag{24}$$

Let $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{q}_1$. If $\boldsymbol{r} = \boldsymbol{0}$, then $\boldsymbol{q} = \boldsymbol{q}_1 \in \mathcal{R}'$. Otherwise, let $\boldsymbol{r} \neq \boldsymbol{0}$. If $\boldsymbol{r}$ is a ray in $\mathcal{R}$, then it is also a ray in $\mathcal{R}'$ and there exists $\rho \in \mathbb{R}_+$ such that $\boldsymbol{q} = \boldsymbol{q}_1 + \rho\boldsymbol{r} \in \mathcal{R}'$. Suppose therefore that $\boldsymbol{r} \neq \boldsymbol{0}$ is not a ray of $\mathcal{R}$. Then there must exist a minimum value $\rho_2 > 0$ such that, for all $\rho > \rho_2$, we have $\boldsymbol{q}_1 + \rho\boldsymbol{r} \notin \mathcal{R}$.

Thus, let $\boldsymbol{q}_2 = \boldsymbol{q}_1 + \rho_2 \boldsymbol{r} = (\boldsymbol{w}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Note that, as $\rho_2 > 0$, $\boldsymbol{q}_2 \neq \boldsymbol{q}_1$. Thus, by choice of ρ_2, there must exist a constraint $c \in \mathcal{C}$ that saturates $\boldsymbol{q}_2$ but not $\boldsymbol{q}_1$. Since no constraint in $\mathcal{C}_\epsilon$ can be saturated by $\boldsymbol{q}_2$, we have $c \notin \mathcal{C}_\epsilon$. Suppose that $c \in \mathcal{C}_>$. Then, by Lemma 12, $\boldsymbol{q}_2 \in \mathrm{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$; since $(\mathcal{G}_R, \mathcal{G}_C) \sqsubseteq \mathcal{G}'$, we obtain $\boldsymbol{q}_2 \in \mathcal{R}'$. Suppose now that $c \in \mathcal{C}_\geq$; then, as $c \notin \mathrm{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_\geq)$, by (24), we obtain $c \notin \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq)$. Similarly, supposing now $c \in \mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon)$, we obtain again $c \notin \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C})$, because otherwise we would have $\boldsymbol{p}_0 \notin \mathcal{R}$. In both cases, as $\boldsymbol{q}_2$ saturates constraint c, then $\boldsymbol{q}_2$ can be obtained as a combination of generators in $\mathcal{G}$ all of which saturate c, i.e., a combination where $\boldsymbol{p}$ has a zero coefficient, so that $\boldsymbol{q}_2 \in \mathrm{gen}(\mathcal{G}') = \mathcal{R}'$.

Thus, in all cases $\boldsymbol{q}_2 \in \mathcal{R}'$ so that, as $\boldsymbol{q}$ lies on the segment $[\boldsymbol{q}_1, \boldsymbol{q}_2]$ and $\mathcal{R}'$ is a convex set, we have $\boldsymbol{q} \in \mathcal{R}'$ as required. □

Lemma 17. *Let* $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. *Let* $\mathcal{R} \in \mathbb{CP}_{n+1}$ *be such that* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ *is a DD pair in minimal form and* $\mathcal{R} \Rrightarrow_Y \mathcal{P}$. *If* $\boldsymbol{p}$ *is* ϵ*-redundant in* $\mathcal{G} = (R, P)$, *then* $\boldsymbol{p}$ *is unmatched in* $\mathcal{G}$ *and* $\mathrm{gen}(\mathcal{G}') \Rrightarrow_Y \mathcal{P}$, *where* $\mathcal{G}' = \bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)$.

Proof. Let $\mathcal{R}' = \mathrm{gen}(\mathcal{G}')$ and $P' = P \setminus \{\boldsymbol{p}\}$, so that $\mathcal{G}' = (R, P')$. Note that $\mathcal{G}' \sqsubseteq \mathcal{G}$ and hence, as the function 'gen' is monotonic, $\mathcal{R}' \subseteq \mathcal{R}$. Also note that any ray in $\mathcal{R}$ is also a ray in $\mathcal{R}'$.

Suppose that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ is ϵ-redundant in $\mathcal{G}$ so that, by Definition 9, $\boldsymbol{p} \in P$, $e > 0$ and there exists a point $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e')^{\mathrm{T}}$ such that $\boldsymbol{p}' \in P'$, $e' > 0$ and

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}). \tag{25}$$

Note that $\boldsymbol{p}' \in \mathcal{R}'$. By Lemma 15, $\boldsymbol{p}$ is unmatched in $\mathcal{G}$. Letting $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and $\boldsymbol{p}_0' = (\boldsymbol{y}^{\mathrm{T}}, 0)^{\mathrm{T}}$, by condition (3) of Definition 2, we have $\{\boldsymbol{p}_0, \boldsymbol{p}_0'\} \subseteq \mathcal{R}$.

As (25) holds, we can use Lemma 16, to obtain that, for all $\boldsymbol{w} \in \mathbb{R}^n$,

$$(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \iff (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'. \tag{26}$$

In order to show that $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$, we first prove that $\mathcal{R}'$ is an ϵ-polyhedron. Consider condition (2) of Definition 2. As $\mathcal{R}' \subseteq \mathcal{R}$, $\mathcal{R}'$ satisfies condition (2) by taking the same value δ used for $\mathcal{R}$. Consider now condition (3) of Definition 2. Let $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}'$. Since $\mathcal{R}' \subseteq \mathcal{R}$, we have $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$; since $\mathcal{R}$ is an ϵ-polyhedron, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then, by (26), we obtain $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$.

Thus, $\mathcal{R}'$ is an ϵ-polyhedron. To show that it is indeed an ϵ-polyhedron for $\mathcal{P}$, we have to prove that $[\![\mathcal{R}]\!] = [\![\mathcal{R}']\!]$. The inclusion $[\![\mathcal{R}']\!] \subseteq [\![\mathcal{R}]\!]$ holds by monotonicity of function $[\![\cdot]\!]$, since $\mathcal{R}' \subseteq \mathcal{R}$. To prove the other inclusion, suppose that $\boldsymbol{w} \in [\![\mathcal{R}]\!]$. Then there exists $e_w > 0$ such that $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$. By condition (3) of Definition 2, we obtain $\boldsymbol{q}_0 = (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and hence, by (26), $\boldsymbol{q}_0 \in \mathcal{R}'$. As $\boldsymbol{q} \in \mathcal{R}$, by definition of 'gen' there exist $0 \leq \pi \leq 1$ and $\boldsymbol{p}_1 \in \mathrm{gen}(\mathcal{G}')$ such that $\boldsymbol{q} = \pi \boldsymbol{p} + (1-\pi)\boldsymbol{p}_1$. If $\pi = 0$, then $\boldsymbol{q} \in \mathcal{R}'$, so that $\boldsymbol{w} \in [\![\mathcal{R}']\!]$ as required. Suppose now that $\pi > 0$; then $\mathrm{sat\_con}(\boldsymbol{q}, \mathcal{C}) \subseteq \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C})$. Thus, by (25), we obtain

$$\mathrm{sat\_con}(\boldsymbol{q}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}).$$

By applying Lemma 10 twice, we obtain

$$\begin{aligned}\operatorname{sat\_con}\bigl(\boldsymbol{q}_0, \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon\bigr) &= \operatorname{sat\_con}(\boldsymbol{q}, \mathcal{C}_\geq) \\ &\subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}_\geq) \\ &= \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon) \\ &\subseteq \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}).\end{aligned}$$

Now let $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\bigr)$ be such that $c \in \mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon$. Then, as $\boldsymbol{p}' \in \mathcal{R}$, $\langle \boldsymbol{a}, \boldsymbol{y}\rangle \geq b$. If $c \in \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$, then $c \in \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C})$, so that all the points lying on the line passing through $\boldsymbol{q}_0$ and $\boldsymbol{p}'_0$ saturate c. On the other hand, if $c \notin \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$, then $\langle \boldsymbol{a}, \boldsymbol{w}\rangle > b$. Let

$$\begin{aligned}\rho_c &= \begin{cases} \frac{\langle \boldsymbol{a}, \boldsymbol{y}\rangle - \langle \boldsymbol{a}, \boldsymbol{w}\rangle}{\langle \boldsymbol{a}, \boldsymbol{y}\rangle - b} & \text{if } \langle \boldsymbol{a}, \boldsymbol{y}\rangle > \langle \boldsymbol{a}, \boldsymbol{w}\rangle; \\ 1 & \text{otherwise;}\end{cases} \\ \boldsymbol{q}_c &= (1 + \rho_c)\boldsymbol{q}_0 - \rho_c \boldsymbol{p}'_0.\end{aligned}$$

Thus $\boldsymbol{q}_c = (\boldsymbol{w}_c^{\mathrm{T}}, 0)^{\mathrm{T}}$ is an affine combination of $\boldsymbol{q}_0$ and $\boldsymbol{p}'_0$, therefore lying on the line passing through these two points. Note that $\rho_c > 0$ and $\boldsymbol{q}_c$ satisfies constraint c. Let c vary in the set of constraints $\mathcal{C} \setminus \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$ and take $\rho > 0$ to be the minimum of all the ρ_c obtained as above. Consider the affine combination

$$\boldsymbol{q}_\rho = (1 + \rho)\boldsymbol{q}_0 - \rho \boldsymbol{p}'_0.$$

Then $\boldsymbol{q}_\rho = (\boldsymbol{w}_\rho^{\mathrm{T}}, 0)^{\mathrm{T}}$ satisfies all the constraints in $\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon$. By Lemma 9, $\boldsymbol{q}_\rho$ also satisfies every constraint in $\mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geq \cup \mathcal{C}_\epsilon)$, so that $\boldsymbol{q}_\rho \in \mathcal{R}$. Thus, by (26), $\boldsymbol{q}_\rho \in \mathcal{R}'$. Letting $\sigma = \frac{1}{1+\rho}$ we obtain $0 < \sigma < 1$ and $\boldsymbol{w} = \sigma \boldsymbol{w}_\rho + (1 - \sigma)\boldsymbol{y}$. Thus, we have

$$\bigl(\boldsymbol{w}^{\mathrm{T}}, (1 - \sigma)e'\bigr)^{\mathrm{T}} = \sigma \boldsymbol{q}_\rho + (1 - \sigma)\boldsymbol{p}' \in \mathcal{R}'.$$

As $e' > 0$ and $\sigma < 1$, we have $(1 - \sigma)e' > 0$ and hence $\boldsymbol{w} \in [\![\mathcal{R}']\!]$ as required.

Suppose next that $\mathcal{R} \Rrightarrow_C \mathcal{P}$. Then, by the first part of the proof $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. By Definition 4, $\mathcal{R} \subseteq \operatorname{con}\bigl(\{\epsilon \geq 0\}\bigr)$. Thus, if $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \cup P$, $e \geq 0$. Since $\mathcal{G}' \sqsubseteq \mathcal{G}$, we also obtain $\operatorname{gen}(\mathcal{G}') \subseteq \operatorname{con}\bigl(\{\epsilon \geq 0\}\bigr)$, so that $\mathcal{R}' \Rrightarrow_C \mathcal{P}$.

Finally, suppose that $\mathcal{R} \Rrightarrow_G \mathcal{P}$. By the first part of the proof $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. Since $\mathcal{R} \neq \varnothing$, by Definition 6, we have that $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}$. Thus, $-\boldsymbol{e}_\epsilon$ is also a ray of $\mathcal{R}'$ and $\mathcal{R}' \Rrightarrow_G \mathcal{P}$, as required. □

Proof (Proof of Proposition 6 on page 12). Items 1 and 2 have been proved as Lemmas 14 and 17 respectively. □

The proof of Proposition 7 is based on the following lemmas.

Lemma 18. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ is a DD pair in minimal form and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. If $\mathcal{C}$ has no ϵ-redundant constraint then it is in* smf.

Proof. To prove the thesis, we assume that $\mathcal{C}$ is not in smf and show that $\mathcal{C}$ must contain an ϵ-redundant constraint. By Proposition 2, $\mathcal{P} = \operatorname{con}\bigl(\operatorname{con\_enc}(\mathcal{C})\bigr)$. As

$\mathcal{C}$ is not in smf, by Definition 8, there exists a constraint system $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con\_enc}(\mathcal{C}') \subset \mathrm{con\_enc}(\mathcal{C})$ and $\mathrm{con}\bigl(\mathcal{C}' \cup \{\epsilon \leq 1\}\bigr) \Rrightarrow_\epsilon \mathcal{P}$. Let $\mathcal{R}' = \mathrm{con}(\mathcal{C}')$ and $\mathcal{R}'_1 = \mathrm{con}\bigl(\mathcal{C}' \cup \{\epsilon \leq 1\}\bigr)$.

Suppose that all the constraints in $\mathrm{con\_enc}(\mathcal{C}) \setminus \mathrm{con\_enc}(\mathcal{C}')$ are non-strict inequalities. Then, by Definition 3, $\mathrm{con\_enc}(\mathcal{C}_>) = \mathrm{con\_enc}(\mathcal{C}'_>)$. Also, by Definition 3, $\mathrm{con\_enc}(\mathcal{C}) = \mathrm{con\_enc}(\mathcal{C}_> \cup \mathcal{C}_\geq)$ and $\mathrm{con\_enc}(\mathcal{C}') = \mathrm{con\_enc}(\mathcal{C}'_> \cup \mathcal{C}'_\geq)$; so that, as $\mathrm{con\_enc}(\mathcal{C}') \neq \mathrm{con\_enc}(\mathcal{C})$, $\mathcal{C}'_\geq \neq \mathcal{C}_\geq$. Thus, since $\mathcal{C}' \subseteq \mathcal{C}$, we obtain $\mathcal{C}'_\geq \subset \mathcal{C}_\geq$. Therefore there exist $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$ such that

$$c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C} \setminus \mathcal{C}'.$$

Let $\mathcal{R}_0 = \mathrm{con}\bigl(\mathcal{C} \setminus \{c\}\bigr)$. Then, as $\mathcal{C}' \subseteq \mathcal{C} \setminus \{c\} \subset \mathcal{C}$, $\mathcal{R}_0 \subseteq \mathcal{R}'$ and $\mathcal{R} \subseteq \mathcal{R}_0$. We now show that $\mathcal{R}_0 \subseteq \mathcal{R}$ so that $\mathcal{R} = \mathcal{R}_0$; therefore contradicting the hypothesis that $\mathcal{C}$ is in minimal form. For this, consider any vector $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_0$. We need to show that $\boldsymbol{p} \in \mathcal{R}$. Since $\mathcal{R}_0 \subseteq \mathcal{R}'$, $\boldsymbol{p} \in \mathcal{R}'$. We prove that $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$ and we do this by considering the cases $e \leq 1$ and $e > 1$ separately. If $e \leq 1$, then $\boldsymbol{p}$ satisfies $c_u = (\epsilon \leq 1)$ so that $\boldsymbol{p} \in \mathcal{R}'_1$. In this case, as $\mathcal{R}'_1 \Rrightarrow_\epsilon \mathcal{P}$, by condition (3) of Definition 2, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'_1$ and hence, by Lemma 4, $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. Suppose now that $e > 1$; then $\boldsymbol{p} \in \mathrm{con}\bigl(\mathcal{C}'_> \cup \mathcal{C}'_\geq \cup \mathcal{C}'_\epsilon \cup \{c_\ell\}\bigr)$, where $c_\ell = (\epsilon \geq 0)$, and hence, $(\boldsymbol{v}^{\mathrm{T}}, 1)^{\mathrm{T}} \in \mathcal{R}''_1 = \mathrm{con}\bigl(\mathcal{C}'_> \cup \mathcal{C}'_\geq \cup \mathcal{C}'_\epsilon \cup \{c_u, c_\ell\}\bigr)$. Since $\mathcal{R}'_1 \Rrightarrow_\epsilon \mathcal{P}$, it follows from Lemma 9, that $\mathcal{R}''_1 \Rrightarrow_\epsilon \mathcal{P}$; hence, by Definition 2, $\boldsymbol{v} \in \mathcal{P}$. Thus in both cases we have $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. Therefore, as $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, by Lemma 4, we obtain $\boldsymbol{p}_0 \in \mathcal{R}$ so that $\boldsymbol{p}_0$ satisfies c. As $c \in \mathcal{C}_\geq$, $\boldsymbol{p}$ also satisfies c and hence $\boldsymbol{p} \in \mathcal{R}$. As $\boldsymbol{p}$ is an arbitrary point in $\mathcal{R}_0$, $\mathcal{R}_0 \subseteq \mathcal{R}$, as required.

Thus there must exist a constraint $c_1 \in \mathrm{con\_enc}(\mathcal{C}) \setminus \mathrm{con\_enc}(\mathcal{C}')$ which is a strict inequality, so that $c_1 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr)$ for some $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$. By Definition 3, for some $s < 0$,

$$c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \in \mathcal{C} \setminus \mathcal{C}'.$$

We now show that c is ϵ-redundant in $\mathcal{C}$. Letting $\mathcal{G} = (R, P)$, suppose first that no point in $\mathrm{gen}\bigl((R, \mathcal{G}_C)\bigr)$ saturates c; this implies that no point in $\mathcal{G}_C$ saturates c, so that $\mathrm{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr) \sqsubseteq (\mathcal{G}_R, \varnothing)$. It follows from Definition 9 that c is ϵ-redundant in $\mathcal{C}$.

Suppose next that there exists a point $\boldsymbol{p} \in \mathrm{gen}\bigl((R, \mathcal{G}_C)\bigr)$ saturating c and let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$. Note that, if e_c is the value of the ϵ coordinate of a point in $\mathcal{G}_C$, then $e_c = 0$. Also, by Lemma 2, if e_r is the value of the ϵ coordinate of a ray in R, then $e_r \leq 0$. Thus we obtain $e_v \leq 0$. Since $\boldsymbol{p} \in \mathcal{R}$, by condition (3) of Definition 2, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since $\boldsymbol{p}$ saturates c and $s < 0$, we obtain $e_v = 0$, i.e., $\boldsymbol{p} = \boldsymbol{p}_0$. Thus $\boldsymbol{p} \in \mathrm{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$.

We show that there exist $c' \in \mathcal{C}'_>$ that is saturated by $\boldsymbol{p}$. Since $s < 0$, we have $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin \mathcal{R}$, for all $e > 0$. As $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, by Definition 2, $\boldsymbol{v} \notin \mathcal{P}$ and, by Lemma 4, we have $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. By hypothesis, $\mathcal{R}'_1 \Rrightarrow_\epsilon \mathcal{P}$ so that, by applying again Definition 2 and Lemma 4, we obtain $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin \mathcal{R}'_1$, for all $e > 0$, and $\boldsymbol{p} \in \mathcal{R}'_1$. As a consequence, there must exist $c' = \bigl(\langle \boldsymbol{a}', \boldsymbol{x} \rangle + s' \cdot \epsilon \geq b'\bigr) \in \mathcal{C}'$ such that c' is saturated by $\boldsymbol{p}$ but not satisfied by $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$, for any $e > 0$. Thus $s' < 0$ and hence, $c' \in \mathcal{C}'_>$ (note that it cannot be $c' = (\epsilon \leq 0)$ because we have $\mathcal{P} \neq \varnothing$).

Let $\mathcal{H} = (\mathcal{H}_R, \mathcal{H}_C) = \mathrm{sat\_gen}\bigl(c, (\mathcal{G}_R, \mathcal{G}_C)\bigr)$ and $\mathcal{C}'_{>} = \{c'_1, \dots, c'_k\}$. Suppose that, for each $1 \le i \le k$, there exists a point $\boldsymbol{q}_i \in \mathrm{gen}(\mathcal{H})$ that does not saturate c'_i. Then the convex combination $\frac{1}{k} \sum_{i=1}^{k} \boldsymbol{q}_i \in \mathrm{gen}(\mathcal{H})$ saturates c, but does not saturate any constraint in $\mathcal{C}'_{>}$, contradicting the previous paragraph. As a consequence, there exists a constraint $c'_j \in \mathcal{C}'_{>}$ that is saturated by all the points in $\mathrm{gen}(\mathcal{H})$, so that $\mathcal{H} \sqsubseteq \mathrm{sat\_gen}(c'_j, \mathcal{G})$. It then follows from Definition 9 that c is ϵ-redundant in $\mathcal{C}$. □

Lemma 19. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ is a DD pair in minimal form and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. If $\mathcal{G}$ has no ϵ-redundant generator then it is in* smf*.*

Proof. To prove the thesis, we assume that $\mathcal{G}$ is not in smf and show that $\mathcal{G}$ must have an ϵ-redundant generator. Let $(R, P) = \mathcal{G}$ and $(R_1, P_1, C_1) = \mathrm{gen\_enc}(\mathcal{G})$. Since $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$ then, by Proposition 2, we have $\mathcal{P} = \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G})\bigr)$. As $\mathcal{G}$ is in minimal form but not in smf, then by Definition 8, there exists $\mathcal{G}' = (R', P')$ such that $\mathcal{G}' \sqsubset \mathcal{G}$, $\mathcal{R}' = \mathrm{gen}(\mathcal{G}') \Rrightarrow_\epsilon \mathcal{P}$ and $\mathrm{gen\_enc}(\mathcal{G}') \sqsubset \mathrm{gen\_enc}(\mathcal{G})$. Let $(R'_1, P'_1, C'_1) = \mathrm{gen\_enc}(\mathcal{G}')$; thus we have $R'_1 \subset R_1$, or $P'_1 \subset P_1$, or $C'_1 \subset C_1$. Note that, again by Proposition 2, we have $\mathcal{P} = \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G}')\bigr)$.

We first suppose that there exists $\boldsymbol{v} \in R_1 \setminus R'_1$ and derive a contradiction. By Definition 3, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R \setminus R'$. Thus $\boldsymbol{v}$ is a ray in $\mathcal{P}$ and hence $\boldsymbol{v}$ is the positive combination of rays in R'_1. By Definition 3, there is a ray $\boldsymbol{r} \in R'_1$ if and only if there is a ray $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R'$. Therefore $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is the positive combination of rays in $R \setminus \bigl\{(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\}$; contradicting the hypothesis that $\mathcal{G}$ is in minimal form.

We next suppose there exists $\boldsymbol{v} \in C_1 \setminus C'_1$ and derive a contradiction. By Definition 3, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P$ and $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin P$ for all $e > 0$. As $P' \subseteq P$, we also have that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin P'$ for all $e > 0$. As a consequence, as $\boldsymbol{v} \notin C'_1$, by Definition 3 we obtain $\boldsymbol{p}_0 \notin P'$. Since $\mathcal{P} \neq \varnothing$, we have $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. Thus, since $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$, we can apply Lemma 4 to obtain $\boldsymbol{p}_0 \in \mathcal{R}'$. Let $\mathcal{G}'' = \bigl(R, P \setminus \{\boldsymbol{p}_0\}\bigr)$; then, we have $\mathcal{G}' \sqsubseteq \mathcal{G}'' \sqsubset \mathcal{G}$. By the monotonicity of the function gen, we obtain $\boldsymbol{p}_0 \in \mathrm{gen}(\mathcal{G}'')$, so that $\mathrm{gen}(\mathcal{G}'') = \mathcal{R}$, contradicting the hypothesis that $\mathcal{G}$ is in minimal form.

Therefore there exists a vector $\boldsymbol{v} \in P_1 \setminus P'_1$. By Definition 3, there exists $e > 0$ such that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P$, so that $\boldsymbol{p} \in \mathcal{G}_P$. Moreover, since $\boldsymbol{v} \notin P'_1$, we have $\boldsymbol{p} \notin \mathcal{G}'_P$. Let $\mathcal{C}' = \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geq})$ and

$$\mathcal{C}'_1 = \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}' \Bigr\}.$$

(Note that the constraints in $\mathcal{C}'$ are defined on the vector space $\mathbb{R}^{n+1}$, whereas those in $\mathcal{C}'_1$ are defined on $\mathbb{R}^n$.) Then $\boldsymbol{v}$ saturates all the constraints in $\mathcal{C}'_1$. For all $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\bigr) \in \mathcal{C}'$ and all $s < 0$, we have $c_s = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\bigr) \notin \mathcal{C}$, because otherwise we would have $\boldsymbol{p} \notin \mathcal{R}$. Thus, by Definition 3, we obtain $\mathcal{C}'_1 \subseteq \mathrm{con\_enc}(\mathcal{C})$, which also implies $\mathcal{P} \subseteq \mathrm{con}(\mathcal{C}'_1)$.

As $\mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G}')\bigr) = \mathcal{P}$, then $\boldsymbol{v}$ can be obtained by combining the generators in (R'_1, P'_1, C'_1). By definition of gen, this means that there exists a point $\boldsymbol{y} \in P'_1$ that, in such a combination, has a strictly positive coefficient; this implies that $\mathrm{sat\_con}\bigl(\boldsymbol{v}, \mathrm{con\_enc}(\mathcal{C})\bigr) \subseteq \mathrm{sat\_con}\bigl(\boldsymbol{y}, \mathrm{con\_enc}(\mathcal{C})\bigr)$; in particular, since $\mathcal{C}_1' \subseteq \text{con\_enc}(\mathcal{C})$, we obtain $\text{sat\_con}(\boldsymbol{v}, \mathcal{C}_1') \subseteq \text{sat\_con}(\boldsymbol{y}, \mathcal{C}_1')$. By Definition 3, there exists $e' > 0$ such that $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e')^{\mathrm{T}} \in P'$, so that $\boldsymbol{p}' \in \mathcal{G}_P'$. Since $\mathcal{G}' \sqsubset \mathcal{G}$, we also obtain $\boldsymbol{p}' \in \mathcal{G}_P \setminus \{\boldsymbol{p}\}$. Observe that, as $\boldsymbol{y}$ saturates every constraint in $\mathcal{C}_1'$, then $\boldsymbol{p}'$ saturates every constraint in $\mathcal{C}'$. It follows that $\text{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geq}) \subseteq \text{sat\_con}(\boldsymbol{p}', \mathcal{C}_{\geq})$ and, by Definition 9, $\boldsymbol{p}$ is ϵ-redundant in $\mathcal{G}$. □

Proof (Proof of Proposition 7 on page 12). Items 1 and 2 have been proved as Lemmas 18 and 19 respectively. □