Widening Sharing

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Abstract

We study the problem of an efficient *and* precise sharing analysis of (constraint) logic programs. After recognizing that neither plain Sharing nor its non-redundant (but equivalent) abstraction scale well to real programs, we consider the domain proposed by C. Fecht [11, 12]. This domain consists of a combination of *Pos* with a quite weak abstraction of Sharing. While verifying that this domain is truly remarkable, in terms of both precision and efficiency, we have revealed significant precision losses for several real programs. This loss concerns groundness, pair-sharing and linearity. We define a simple domain for sharing analysis that supports the implementation of several widening techniques. In particular, with this domain it is straightforward to turn Fecht's idea into a proper widening. More precise widenings are also considered. However, in spite of thorough experimentation we found that, provided *Pos* is included in the domain, the first widening we propose is hard to improve on.

Keywords: Abstract Interpretation, Mode Analysis, Sharing Analysis, Widening.

1 Introduction

For (constraint) logic programs, the main purpose of sharing analysis is to detect pair-sharing, that is, which *pairs* of variables are definitely independent. In a previous work [3] we observed that the Sharing domain of Jacobs and Langen (denoted here as SH) [14] is redundant for pair-sharing. This achievement has important theoretical consequences (some of which will be exploited in the present work) and also has a practical interest. In fact, it allows sharing-sets (the elements of SH) to be kept as small as possible without any precision loss and replace the *star-union* operation, whose complexity is exponential, by *self-bin-union*, which is quadratic. Even though significant speed-ups have been observed in practice (up to 3 orders of magnitude) the problem of scalability of the analysis to real programs, both in terms of precision (that is, the number of pairs that are detected as being definitely independent) and of resource usage, was still to be solved.

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C. Fecht [11, 12] proposed a domain $\downarrow SH$ for sharing analysis based on SH. This domain is the same as SH but the concretization of a set of variables in $\downarrow SH$ is equivalent to the concretization of its powerset in SH. The advantage of $\downarrow SH$ is not just that an element can be normalized by removing all but the maximal ones, thereby reducing its size, but because it enables more efficient (but less precise) abstract operations than those used for SH and its non-redundant version SH^ρ [3].

One of the problems with the domain $\downarrow SH$ is that it does not capture ground dependencies. These are important for tracking sharing dependencies and, hence, sharing. Fecht solved this by deriving the ground dependencies through the Pos component of the combined domain $Pos + \downarrow SH$ (where Pos is the usual domain of positive Boolean formulas for groundness analysis) and also $Pos + \downarrow SH + Lin$ (where Lin is a domain for representing variables that are definitely bound to linear terms). Fecht tested both these domains and showed that, with his benchmarks, they compared favorably with equivalent ones using SH for the sharing and ground dependencies. He reported a negligible loss of precision and demonstrated that large programs could be analyzed using both $Pos + \downarrow SH$ and $Pos + \downarrow SH + Lin$ in a reasonable time scale. The results, although inconclusive, demonstrated real promise for an analyzer based on the $\downarrow SH$ approach. We say the results were inconclusive. The reason for this is that only a few non-trivial programs were tested and, for most of these, precision was not compared. (His SH analyzer could not cope with large programs possibly due to the problem that there was no redundancy elimination.) We note that Fecht did not present the domain $\downarrow SH$ as a widening and did not discuss how a widening based on his domain might be achieved.

1.1 Precision Results for SH and $\downarrow SH$

We have compared the domain of Fecht enhanced with *Free*, the domain representing *freeness information*, that is $Pos + \downarrow SH + Free + Lin$, with the same combination where $\downarrow SH$ is substituted by the non-redundant sharing domain SH^ρ [3]. The precision of the analysis is measured by summing results over the success-patterns, for goal-independent (GI) analysis, and over both the call- and success-patterns, for goal-dependent (GD) analysis, for each procedure. For the domains tested, that is, $Pos + \downarrow SH + Free + Lin$, abbreviated as P+DSH+F+L, and $Pos + SH^\rho + Free + Lin$, abbreviated as P+NSH+F+L, the precision results consist of: the total number of definitely non-sharing pairs of program variables, NSP, the total number of definitely ground variables, GV, and the total number of definitely linear variables that are possibly not ground, LV. The freeness results are not compared because, as we have shown in [18], freeness is not affected, neither by abstracting SH to $\downarrow SH$, nor by redundancy elimination.

The comparison involved all the 92 Prolog programs in our current test-suite. On 73 of them there was no difference in precision. This is really remarkable considering that the $\downarrow SH$ approximation is rather crude. The combined domain $Pos + \downarrow SH + Lin$ is isomorphic to $\mathsf{ASub} + Pos$ (where ASub is the pair-sharing domain of Søndergaard [17]), and the domain $Pos + \downarrow SH$ is exactly the domain ASub^+ defined by Cortesi and Filé in [8]. However, they considered this domain only *en passant* and only from a theoretical point of view. In other words, Fecht has the whole merit for having

	Goal-Independent						Goal-Dependent					
	P+DSH+F+L			P+NSH+F+L			P+DSH+F+L			P+NSH+F+L		
Program	NSP	GV	LV	NSP	GV	LV	NSP	GV	LV	NSP	GV	LV
aqua_c	10749	⋆406	2753	?	?	?	16306	⋆1186	2028	?	?	?
bmtp	1451	136	972	1461	136	976						
bryant	784	10	146	1088	10	223	1033	141	58	1781	141	58
caslog	6456	⋆466	1588	7027	⋆474	1615	11073	⋆1625	1079	?	?	?
cg_pars	136	31	159	138	31	160						
dpos_an	92	40	76	95	40	76	183	76	53	183	76	53
lg_sys	7274	645	2260	7334	645	2261						
nand	473	23	182	475	23	182	1341	481	70	1341	481	70
nbody	261	52	104	262	52	104	477	151	41	478	151	41
ochina	2185	285	1163	2193	285	1166	3985	802	760	?	?	?
quot_an	288	37	160	288	37	160	639	159	122	646	159	122
reg	774	42	272	796	42	284	207	67	52	207	67	52
rubik	70	⋆55	110	73	⋆76	93	174	⋆110	124	201	⋆200	103
scc	63	0	37	63	0	37	503	174	46	506	174	46
sfecht	28	0	14	85	0	31	221	0	47	278	0	64
simple	370	27	139	373	27	139	572	82	76	639	82	76
slice	427	126	453	428	126	453						
spsys	788	81	386	800	81	394						
trs	32	6	22	53	6	22	73	⋆12	20	104	⋆12	20

Table 1: $Pos + {\downarrow}SH + Free + Lin$ vs $Pos + SH^{\rho} + Free + Lin$: precision.

trusted on this domain from a precision/efficiency perspective. The results for the remaining 19 programs are summarized in Table 1.

The blank entries in the goal-dependent columns are for those programs whose goal-dependent analysis is pointless. This usually happens because the program contains a procedure call to an unknown procedure (e.g., by means of `call/1`). The China analyzer (i.e., our system [1]) promptly recognizes these cases and reverts to a goal-independent analysis. This is one of the reasons why focusing only on goal-dependent analyses is, in our opinion, a mistake. The other reason being that the ability of analyzing libraries once and for all is desirable and, more generally, so is the separate analysis of different program modules, especially in very large projects. Focusing only on goal-independent analyses is the opposite mistake: GD analyses, when possible, are more precise than GI ones. For these reasons, we insist in presenting experimental results for both.

A star symbol ($\star$) in the GV column signifies that one of the widenings we employ on the GER representation of *Pos* fired. This is a widening imposing a limit on the number of ROBDD nodes simultaneously allocated. It makes approximations of the R (ROBDD) component when this limit is reached[1], while retaining full precision on the G (definitely ground variables) and the E (classes of equivalent variables) components [2, 4]. The scarcity of stars in this and the following tables, shows how seldom this widening is actually required.[2]

Apart from `sfecht`, which is a synthetic benchmark designed in order to show

[1] That is, by approximating $x \wedge y$ with x or with y, $x \vee y$ with *true* and so forth.

[2] Indeed, the newest version of China avoids also the widening for the `caslog` program.

that arbitrary precision losses are possible with $\downarrow SH$, Table 1 illustrates how heavy precision penalties can be incurred by $\downarrow SH$ even on real programs. Most notably, for `bryant` we see a precision loss as high as 28% on GI analysis and 42% on GD analysis. In addition, `simple` loses 10% (GD), while `trs` loses 40% (GI) and 30% (GD). Note that, for these programs, the *Pos* widening fires only on the GD analysis of `trs`. The `rubik` program shows an interesting phenomenon: here the *Pos* widening fires incurring a precision loss of exactly 1 ground variable (a critical one indeed), but SH^{ρ} saves the day by recovering the lost groundness information. A similar thing happens for `caslog`. Thus, the widely held opinion (now proved in [7]) that Sharing does not help *Pos* on groundness does not carry through when widenings are considered.

While space limitations do not allow us to report full timing information, we can easily confirm Fecht's claim: the speedup is dramatic. Just a few examples: the fixpoint computation time in seconds for `bmtp`, `caslog`, `lg_sys`, and `spsys` drops from 15.6, 614.7, 735.9, and 2.2, to 0.8, 2.0, 3.3, and 0.6, respectively. All the experiments described in this paper were performed on a PC equipped with an AMD K6@400MHz, 128MB of main memory, and running Linux 2.2.1.

1.2 The Present Work

The objective of this work, after having recognized that Fecht's approach incurs significant precision loss on several real programs, is to improve the state of the art in mode analysis, in general, and sharing analysis in particular.

The present work is not intended for achieving the fastest ever analysis, since high speeds are often achieved at the expense of precision. The real problem, we believe, is how to increase precision yet avoid the concrete effects of exponential complexity. Consider groundness analysis, for instance. The cruder domains do not pose any efficiency problem. In contrast, the more refined domains for groundness, such as *Pos*, work perfectly until you bump into a "nasty" program clause (i.e., with more than, say, fifty variables for which the analyzer knows too little at that point of the analysis). When this happens, *Pos* will exhaust your computer's memory. One would like to have a more *linear*, or *stable* behavior. The right solution, as indicated in [10], is not to revert to the simpler domains. We should use complex domains instead, together with widening (and maybe narrowing) operators. With this technique we can try to limit precision losses to those cases where we cannot afford the refined domains.

We moved from the observation that, when the sharing-sets become large, then they are at the same time heavy to manipulate and, at least for a subset of the variables involved, light as far as information content is concerned. We thus introduce a new representation SH^{W} for set-sharing made of two components. They are both sharing-sets. However, while the second one is interpreted in the usual way, the first component records worst-case sharing assumptions of sets of variables. We define the operations required for the analysis with SH^{W}, and we prove them correct. We also introduce a safe optimization that is very effective in practice.

We then show how SH^{W} supports a variety of widenings. One of those is a simple adaptation of Fecht's idea. Others are much more sophisticated and involve only a limited precision loss. However, in spite of thorough experimentation (of which only

a tiny fraction can be reported here) we found that the first widening we propose is hard to improve on, provided *Pos* is included in the domain. This suggests that what is lost by this widening is mostly constituted by ground dependencies, and these can be recovered (and improved) by the *Pos* component. We show that when *Pos* is not included, a widening based on cliques of sharing pairs is preferred. Since some authors advocate the use of *SH* without coupling it with *Pos* (we do not share this view), this is an important message for them.

In this work as in Fecht's work, the combination of a sharing domain with *Pos* is the simplest possible. For any operation of the analysis, abstract mgu in particular, the *Pos* component is evaluated first. All sharing groups containing at least one variable that is definitely ground according to the resulting *Pos* formula are removed from the sharing component. This combination is made particularly efficient by the ready availability of definite groundness information allowed by the GER representation introduced in [4], where obtaining the set of definitely ground variables (and also the classes of groundness-equivalent variables) is a constant-time operation. Note that more sophisticated combinations are possible [7].

Following several other authors, we observed in [3], that, from a practical point of view, sharing analysis without freeness or linearity does not make sense. Both these properties allow, in a significant proportion of cases, to dispense with costly operations (such as star-union or, better, self-bin-union) increasing the precision of sharing information at the same time, and this with very little overhead. Moreover, freeness is a useful property in itself. For details on how the combination with freeness is realized, we refer the reader to [16, 18]. See [6] for the combination of both freeness and linearity information.

Among the contributions of this paper we would like to stress the following: we present a data-flow analysis for groundness, freeness, pair-sharing, and linearity, with unprecedented levels of precision *and* efficiency. With the implementation described in this paper, the China analyzer is able to honor one of its most important design goals: never crash (e.g., by exhausting all the available memory), always terminate with a correct result and in reasonable time.

The paper is structured as follows: In Section 2 we briefly recall the required notions and notations, even though we assume general acquaintance with the topics of abstract interpretation, sharing analysis and groundness analysis. Section 3 introduces SH^{W} and the abstract operations for this domain. With SH^{W}, it is straightforward to turn Fecht's idea into a proper widening. This is done in Section 4, after the introduction of an infinite family of widenings and a statement of their safety. More precise widenings are also considered. The experimental evaluation of the proposed approach is presented in Section 5. Section 6 concludes with some final remarks. The reader is referred to [18] for full proofs of all the results presented in this paper, and for more material on this subject.

2 Preliminaries

For any set S, $\wp(S)$ denotes the powerset of S and $\#\, S$ is the cardinality of S. A monotone and idempotent self-map $\rho\colon P \to P$ over a poset $\langle P, \preceq \rangle$ is called a *closure*

operator (or *upper closure operator*) if it is also *extensive*, namely $\forall x \in P : x \preceq \rho(x)$. In this paper, we assume there is a fixed and finite set of variables of interest denoted by VI. If t is a first-order term over VI, then $vars(t)$ denotes the set of variables in t. *Bind* denotes the set of equations of the form $x = t$ where $x \in VI$ and t is a first-order term over VI. Note that we do not impose the *occur-check* condition $x \notin vars(t)$, since we have proved in [13] that this is not required to ensure correctness of the operations of SH and its derivatives. The following definitions are a simplification of the standard definitions for the Sharing domain [9, 13, 14] and assume that the set of variables of interest is fixed and finite.

Definition 1 (The *set-sharing* domain SH.) *The set SH is defined as a powerset: $SH \stackrel{\text{def}}{=} \wp(SG)$, where $SG \stackrel{\text{def}}{=} \{ S \in \wp(VI) \mid S \neq \varnothing \}$.*

We now introduce the required abstract operations over SH.

Definition 2 (Some operations over SH.) Projecting *an element of SH onto a subset of VI is performed through the binary function* $\mathrm{proj}\colon SH \times \wp(VI) \to SH$*: if $sh \in SH$ and $V \in \wp(VI)$, then* $\mathrm{proj}(sh, V) \stackrel{\text{def}}{=} \{ S \cap V \mid S \in sh, S \cap V \neq \varnothing \}$.

For each $sh \in SH$ and each $V \in \wp(VI)$, the extraction of the relevant component of sh with respect to V *is encoded by the function* $\mathrm{rel}\colon \wp(VI) \times SH \to SH$ *defined as* $\mathrm{rel}(V, sh) \stackrel{\text{def}}{=} \{ S \in sh \mid S \cap V \neq \varnothing \}$.

For $sh \in SH$ and $V \in \wp(VI)$, the exclusion of the irrelevant component of sh with respect to V *is encoded by the function* $\overline{\mathrm{rel}}\colon \wp(VI) \times SH \to SH$ *defined as* $\overline{\mathrm{rel}}(V, sh) \stackrel{\text{def}}{=} sh \setminus \mathrm{rel}(V, sh)$.

The function $(\cdot)^\star\colon SH \to SH$*, also called* star-union*, is given, for each $sh \in SH$, by* $sh^\star \stackrel{\text{def}}{=} \{ S \in SG \mid \exists n \geq 1 \,.\, \exists T_1, \ldots, T_n \in sh \,.\, S = T_1 \cup \cdots \cup T_n \}$.

For each $sh_1, sh_2 \in SH$, the binary union *function* $\mathrm{bin}\colon SH \times SH \to SH$ *is given by* $\mathrm{bin}(sh_1, sh_2) \stackrel{\text{def}}{=} \{ S_1 \cup S_2 \mid S_1 \in sh_1, S_2 \in sh_2 \}$.

We also use the self-bin-union *function* $\mathrm{sbin}\colon SH \to SH$*, which is given, for each $sh \in SH$, by* $\mathrm{sbin}(sh) \stackrel{\text{def}}{=} \mathrm{bin}(sh, sh)$.

The function amgu *captures the effects of a binding on an SH element. Let $(x = t) \in Bind$, $sh \in SH$, $V_x = \{x\}$, $V_t = vars(t)$, and $V_{xt} = V_x \cup V_t$. Then*

$$\mathrm{amgu}(sh, x = t) \stackrel{\text{def}}{=} \overline{\mathrm{rel}}(V_{xt}, sh) \cup \mathrm{bin}\big(\mathrm{rel}(V_x, sh)^\star, \mathrm{rel}(V_t, sh)^\star\big).$$

The domain SH captures *set-sharing*. However, the property we wish to detect is *pair-sharing* and, for this, it has been shown that SH includes unwanted redundancy [3].

Definition 3 (Redundancy.) *Let $sh \in SH$ and $S \in SG$. S is* redundant for *sh if and only if $\# S > 2$ and* $\mathrm{pairs}(S) = \bigcup \{ \mathrm{pairs}(T) \mid T \in sh, T \subset S \}$ *where* $\mathrm{pairs}(S) \stackrel{\text{def}}{=} \{ P \in \wp(S) \mid \# P = 2 \}$.

Definition 4 (The domain SH^ρ.) *The function $\rho\colon SH \to SH$ is given, for each $sh \in SH$, by $\rho(sh) \stackrel{\text{def}}{=} sh \cup \{ S \in SG \mid S$ is redundant for $sh \}$. Then $SH^\rho \stackrel{\text{def}}{=} \rho(SH) \stackrel{\text{def}}{=} \{ \rho(sh) \mid sh \in SH \}$.*

We use the notation $sh_1 =_\rho sh_2$ and $sh_1 \subseteq_\rho sh_2$ to denote $\rho(sh_1) = \rho(sh_2)$ and $\rho(sh_1) \subseteq \rho(sh_2)$, respectively. The advantage of SH^ρ is that we can replace the star-union operation in the definition of the amgu by self-bin-union without loss of precision [3]. In particular, it is shown:

$$\mathrm{amgu}(sh, x = t) =_\rho \overline{\mathrm{rel}}(V_{xt}, sh) \cup \mathrm{bin}\Big(\mathrm{sbin}\big(\mathrm{rel}(V_x, sh)\big), \mathrm{sbin}\big(\mathrm{rel}(V_t, sh)\big)\Big). \qquad (1)$$

3 A New Representation for Set-Sharing

We introduce here a new representation for set-sharing. It is made up of two components: one is the original set-sharing domain while the other represents all possible subsets of each of its elements and, for this reason, is called a *clique-set*.

Definition 5 (Clique-set.) *A* clique-set *is an element of* CL *and* $CL \stackrel{\text{def}}{=} SH$.

An element of a clique-set is called a *clique*.

Definition 6 (Sharing-sets representation for clique-sets.) *The (overloaded) functions* $\downarrow\colon SG \to SH$ *and* $\downarrow\colon CL \to SH$ *are given, for each* $C \in SG$ *and each* $cl \in CL$, *by* $\downarrow C \stackrel{\text{def}}{=} \wp(C) \setminus \{\varnothing\}$ *and* $\downarrow cl \stackrel{\text{def}}{=} \bigcup_{C \in cl} \downarrow C$. *Observe that* $\downarrow$ *is an upper closure operator over* SH. *If* $cl \in CL$ *and* $C \in SG$ *then we say that* C *is* down-redundant *in* cl *if there exists* $C' \in cl$ *such that* $C \subset C'$.

The addition or removal of down-redundant elements to or from a clique-set makes no difference to the sharing-sets that it represents. So, a clique represents a *worst case*[3] pair-sharing condition on the set of variables it contains. In an implementation, as we need to keep the clique sets as small as possible, down-redundant elements are removed via a *normalization* function.

Definition 7 (Normalization of Clique-Sets.) *For each* $cl \in CL$, *the normalization function* $|\cdot|\colon CL \to CL$ *is given by*

$$|cl| \stackrel{\text{def}}{=} cl \setminus \{\, C \in cl \mid C \text{ is down-redundant for } cl \,\}.$$

We now define abstract unification over clique-sets and state its soundness.

Definition 8 (Abstract Unification over CL**.)** *For each* $V \in \wp(VI)$ *and each* $cl \in CL$, *the function* $\overline{\mathrm{rel}}^{\mathsf{CL}}\colon \wp(VI) \times CL \to CL$ *is given by*

$$\overline{\mathrm{rel}}^{\mathsf{CL}}(V, cl) \stackrel{\text{def}}{=} \{\, C \setminus V \mid C \in cl \,\} \setminus \{\varnothing\}.$$

The function $\mathrm{amgu}^{\mathsf{CL}}\colon CL \times Bind \to CL$ *is given, for each* $cl \in CL$ *and each binding* $(x = t) \in Bind$, *by*

$$\mathrm{amgu}^{\mathsf{CL}}(cl, x = t) \stackrel{\text{def}}{=} \overline{\mathrm{rel}}^{\mathsf{CL}}(V_{xt}, cl) \cup \mathrm{bin}\big(\mathrm{sbin}(cl_x), \mathrm{sbin}(cl_t)\big),$$

where $cl_x = \mathrm{rel}(V_x, cl)$, $cl_t = \mathrm{rel}(V_t, cl)$, $V_x = \{x\}$, $V_t = vars(t)$, *and* $V_{xt} = V_x \cup V_t$.

[3] While this terminology is due to Langen [15], our definition differs from the one he used.

Theorem 9 *For each clique-set $cl \in CL$ and each binding $(x = t) \in Bind$, we have that* $\mathrm{amgu}\big(\downarrow cl, x = t\big) \subseteq_\rho \downarrow \mathrm{amgu}^{\mathsf{CL}}(cl, x = t)$.

We next define our new sharing domain for widening.

Definition 10 (The SH^{W} Representation.) *The set SH^{W} is given by*

$$SH^{\mathsf{W}} \stackrel{\text{def}}{=} \big\{\ (cl, sh)\ \big|\ cl \in CL, sh \in SH\ \big\}$$

and is ordered by $\sqsubseteq$ defined as follows, for each $shw, (cl_1, sh_1), (cl_2, sh_2) \in SH^{\mathsf{W}}$: $(cl_1, sh_1) \sqsubseteq (cl_2, sh_2) \iff (cl_1 \subseteq cl_2) \wedge (sh_1 \subseteq sh_2)$. It can be seen that SH^{W} is a complete lattice. The sharing-set represented by an element of SH^{W} is given, for each $(cl, sh) \in SH^{\mathsf{W}}$, by the function $\mathcal{I}(\cdot)\colon SH^{\mathsf{W}} \to SH$ where

$$\mathcal{I}\big((cl, sh)\big) \stackrel{\text{def}}{=} \downarrow cl \cup sh.$$

The normalization of an element of SH^{W} is given, for each $(cl, sh) \in SH^{\mathsf{W}}$, by the function $|\cdot|\colon SH^{\mathsf{W}} \to SH^{\mathsf{W}}$ where

$$|(cl, sh)| \stackrel{\text{def}}{=} \big(|cl|, sh \setminus \downarrow cl\big).$$

The normalization removes unnecessary elements from a description in SH^{W}. We now define an upper closure operator ϱ inducing an equivalence relation on the elements of SH^{W}.

Definition 11 (The $\varrho(SH^{\mathsf{W}})$ domain.) *The function $\varrho\colon SH^{\mathsf{W}} \to SH^{\mathsf{W}}$ is given, for each $shw \in SH^{\mathsf{W}}$ with $shw \stackrel{\text{def}}{=} (cl, sh)$, by*

$$\varrho(shw) \stackrel{\text{def}}{=} \big(\rho(\downarrow cl), \rho\big(\mathcal{I}(shw)\big)\big).$$

Then ϱ is an upper closure operator for SH^{W} [18]. We will use the notation $shw_1 =_\varrho shw_2$ to denote $\varrho(shw_1) = \varrho(shw_2)$ and $shw_1 \sqsubseteq_\varrho shw_2$ to denote $\varrho(shw_1) \sqsubseteq \varrho(shw_2)$.

The ordering $\sqsubseteq_\varrho$ is used for modeling the relative precision between widenings in Section 4. When $shw_1 =_\varrho shw_2$, shw_1 and shw_2 behave the same way as far as representing pair-sharing and groundness is concerned.

Proposition 12 *If $shw \in SH^{\mathsf{W}}$, then $\mathcal{I}(shw) =_\rho \mathcal{I}\big(\varrho(shw)\big)$ and $shw =_\varrho |shw|$.*

Definition 13 (Operations over SH^{W}.) *For each $(cl, sh), (cl_i, sh_i) \in SH^{\mathsf{W}}$, with $i = 1, 2$, and each $V \in \wp(VI)$, the functions $\mathrm{rel}^{\mathsf{W}}, \overline{\mathrm{rel}}^{\mathsf{W}}\colon \wp(VI) \times SH^{\mathsf{W}} \to SH^{\mathsf{W}}$ and $\cup^{\mathsf{W}}, \mathrm{bin}^{\mathsf{W}}\colon SH^{\mathsf{W}} \times SH^{\mathsf{W}} \to SH^{\mathsf{W}}$, the functions $\mathrm{sbin}^{\mathsf{W}}\colon SH^{\mathsf{W}} \to SH^{\mathsf{W}}$ and, finally,*

$\mathrm{amgu}^{\mathsf{w}}$: *$SH^{\mathsf{w}} \times Bind \to SH^{\mathsf{w}}$, are defined as follows:*

$$
\begin{aligned}
\mathrm{rel}^{\mathsf{w}}\big(V,(cl,sh)\big) &\stackrel{\text{def}}{=} \big(\mathrm{rel}(V,cl),\mathrm{rel}(V,sh)\big),\\
\overline{\mathrm{rel}}^{\mathsf{w}}\big(V,(cl,sh)\big) &\stackrel{\text{def}}{=} \big(\overline{\mathrm{rel}}^{\mathsf{cL}}(V,cl),\overline{\mathrm{rel}}(V,sh)\big),\\
(cl_1,sh_1)\cup^{\mathsf{w}}(cl_2,sh_2) &\stackrel{\text{def}}{=} \big(cl_1\cup cl_2, sh_1\cup sh_2\big),\\
\mathrm{bin}^{\mathsf{w}}\big((cl_1,sh_1),(cl_2,sh_2)\big) &\stackrel{\text{def}}{=} \big(\mathrm{bin}(cl_1,cl_2)\cup\mathrm{bin}(cl_1,sh_2)\cup\mathrm{bin}(sh_1,cl_2),\\
&\qquad \mathrm{bin}(sh_1,sh_2)\big),\\
\mathrm{sbin}^{\mathsf{w}}\big((cl,sh)\big) &\stackrel{\text{def}}{=} \mathrm{bin}^{\mathsf{w}}\big((cl,sh),(cl,sh)\big)\\
&= \big(\mathrm{sbin}(cl)\cup\mathrm{bin}(cl,sh),\mathrm{sbin}(sh)\big),\\
\mathrm{amgu}^{\mathsf{w}}(shw,x=t) &\stackrel{\text{def}}{=} \overline{\mathrm{rel}}^{\mathsf{w}}(V_{xt},shw)\\
&\qquad \cup^{\mathsf{w}}\mathrm{bin}^{\mathsf{w}}\Big(\mathrm{sbin}^{\mathsf{w}}\big(\mathrm{rel}^{\mathsf{w}}(V_x,shw)\big),\\
&\qquad\qquad \mathrm{sbin}^{\mathsf{w}}(\mathrm{rel}^{\mathsf{w}}(V_t,shw))\Big),
\end{aligned}
$$

where $V_x=\{x\}$, $V_t=vars(t)$, and $V_{xt}=V_x\cup V_t$.

The next theorem, proven in [18], states the correctness of $\mathrm{amgu}^{\mathsf{w}}$ and that normalization does not affect the correctness or precision of $\mathrm{amgu}^{\mathsf{w}}$.

Theorem 14 *For each $shw\in SH^{\mathsf{w}}$ and each $(x=t)\in Bind$,*

$$
\begin{aligned}
\mathrm{amgu}\big(\mathcal{I}(shw),x=t\big) &\subseteq_{\rho} \mathcal{I}\big(\mathrm{amgu}^{\mathsf{w}}(shw,x=t)\big);\\
\mathrm{amgu}^{\mathsf{w}}(shw,x=t) &=_{\varrho} \mathrm{amgu}^{\mathsf{w}}\big(|shw|,x=t\big).
\end{aligned}
$$

In general, ϱ is not a congruence for $\mathrm{amgu}^{\mathsf{w}}$ and precision may be lost when the first component of shw is non-empty and the second component of shw contains redundant elements. Further work on this aspect is ongoing.

The computation of $\mathrm{amgu}^{\mathsf{w}}$ can be optimized. To explain this, we need some extra notation. Let us define: $shw=(cl,sh)\in SH^{\mathsf{w}}$ and $x=t\in Bind$; $V_x=\{x\}$, $V_t=vars(t)$, and $V_{xt}=V_x\cup V_t$; $shw_x=(cl_x,sh_x)=\mathrm{rel}^{\mathsf{w}}(V_x,shw)$ and $shw_t=(cl_t,sh_t)=\mathrm{rel}^{\mathsf{w}}(V_t,shw)$. Suppose $C_x=\bigcup cl_x$, $C_t=\bigcup cl_t$, $S_x=\bigcup sh_x$, $S_t=\bigcup sh_t$, $A_{xt}=\mathrm{bin}\big(\mathrm{sbin}(sh_x),\mathrm{sbin}(sh_t)\big)$, $B_{xt}=\mathrm{bin}(sh_x,sh_t)$, $B_x=\mathrm{bin}(cl_x,sh_x)$, and, finally, $B_t=\mathrm{bin}(cl_t,sh_t)$. Then let

$$
shw^{\mathrm{opt}}_{\mathrm{rel}} \stackrel{\text{def}}{=} \begin{cases}
\big(\{C_x\cup C_t\cup S_x\cup S_t\},\varnothing\big), & \text{if } cl_x\neq\varnothing, cl_t\neq\varnothing;\\
\big(\{C_x\cup S_t\}\cup B_x\cup B_{xt},A_{xt}\big), & \text{if } cl_x\neq\varnothing, cl_t=\varnothing;\\
\big(\{C_t\cup S_x\}\cup B_t\cup B_{xt},A_{xt}\big), & \text{if } cl_x=\varnothing, cl_t\neq\varnothing;\\
(\varnothing,A_{xt}), & \text{if } cl_x=\varnothing, cl_t=\varnothing.
\end{cases}
$$

In [18], we prove the following optimization for the computation of $\mathrm{amgu}^{\mathsf{w}}$.

Theorem 15 *Assuming the above notation,*

$$
\begin{aligned}
&\mathrm{amgu}^{\mathsf{w}}(shw,x=t)\\
&\quad =_{\varrho} \begin{cases}
\big(\overline{\mathrm{rel}}(V_{xt},cl),\overline{\mathrm{rel}}(V_{xt},sh)\big)\cup^{\mathsf{w}} shw^{\mathrm{opt}}_{\mathrm{rel}}, & \textit{if } shw_x\neq(\varnothing,\varnothing), shw_t\neq(\varnothing,\varnothing);\\
\overline{\mathrm{rel}}^{\mathsf{w}}(V_{xt},shw), & \textit{otherwise.}
\end{cases}
\end{aligned}
$$

In Eq. (1), the basic amgu operation is defined using the SH^ρ domain. However, when we have freeness and linearity information it has been proven that we can improve the analysis with respect to precision and efficiency by avoiding one or both of the self-bin-unions occurring as components of the binary union operation. The question arises as to whether this improvement can be adapted for the amgu$^{\mathsf{w}}$ operation. That is, can we avoid the corresponding sbin$^{\mathsf{w}}$ operations under the same linearity and freeness conditions? The answer is *yes*, we can generalize Theorem 14 and show that this is sound. However, we may lose precision. Moreover, the optimization given by Theorem 15 only applies to the basic amgu$^{\mathsf{w}}$ operation given in Definition 13. When one or both of the self-bin-unions here is omitted due to available freeness and linearity information, then $=_\varrho$ in the theorem becomes $\sqsubseteq_\varrho$ and we may lose further precision. Further work on this subject is ongoing.

4 Widening Set-Sharing

We can now define a family of *unary* widenings over SH^{w}.

Definition 16 (Widening for SH^{w}.) *A function $\nabla\colon SH^{\mathsf{w}} \to SH^{\mathsf{w}}$ is a* widening for SH^{w} *if, for each $shw \in SH^{\mathsf{w}}$, we have $shw \sqsubseteq_\varrho \nabla\, shw$.*

The following result establishes the safety of such widening operators.

Theorem 17 *For each $shw \in SH^{\mathsf{w}}$ and each $(x = t) \in Bind$ we have*

$$\mathrm{amgu}^{\mathsf{w}}(shw, x = t) \sqsubseteq_\varrho \mathrm{amgu}^{\mathsf{w}}\big(\nabla\, shw, x = t\big).$$

The obvious corollary is that any analysis using these widenings, possibly a different widening at each step of the analysis, is correct. After widening we usually normalize the new domain to provide a smaller representation. Moreover, it is also shown in [18] that similar results hold for each of the component operators such as bin$^{\mathsf{w}}$ for amgu$^{\mathsf{w}}$. Thus we can (and do) safely widen and normalize within the actual computation of amgu$^{\mathsf{w}}$. The analyzer has the freedom of using whichever widening suits its current needs. Those needs can be dictated by a number of heuristics. Of course, really useful widenings are *guarded* by some applicability condition. The simplest conditions are those based on the cardinality of the sets in the SH^{w} description. For example, for each widening ∇ and for suitable choices of $f\colon \mathbb{N}^2 \to \mathbb{N}$ and $n \in \mathbb{N}$, one can define

$$\nabla_{f,n}(cl, sh) \stackrel{\text{def}}{=} \begin{cases} \nabla(cl, sh), & \text{if } f(\#\, cl, \#\, sh) > n; \\ (cl, sh), & \text{otherwise.} \end{cases}$$

We order the widenings in the obvious way. If ∇_1 and ∇_2 are two widenings and, for all $shw \in SH^{\mathsf{w}}$, $\nabla_1(shw) \sqsubseteq_\varrho \nabla_2(shw)$, then let $\nabla_1 \sqsubseteq_\varrho \nabla_2$. At the top end of the scale of widenings we have two *panic widenings*. They are defined by

$$\nabla^p(cl, sh) \stackrel{\text{def}}{=} \Big(cl \cup \big\{\bigcup sh\big\}, \varnothing\Big), \qquad \nabla^P(cl, sh) \stackrel{\text{def}}{=} \Big(\big\{\bigcup cl \cup \bigcup sh\big\}, \varnothing\Big).$$

The panic widenings are present in the China implementation, with very strict guards, only to enforce the "never crash" motto: no real program we have access to makes them fire. At the other extreme we have very soft widenings.

Definition 18 (Cautious widening.) *A widening* $\nabla : SH^{\mathrm{W}} \to SH^{\mathrm{W}}$ *is called* cautious *if, for each* $shw \in SH^{\mathrm{W}}$,

$$\mathcal{I}(\nabla\, shw) =_\rho \mathcal{I}(shw).$$

Thus, a widening never introduces new pair-sharings nor new singletons in the description. However, information is lost as soon as the operations for the analysis given by Definition 13 are considered. For example, consider two elements of SH^{W}: $shw_1 \stackrel{\text{def}}{=} \big(\varnothing, \{x, y, z, xy, xz, yz\}\big)$ and $shw_2 \stackrel{\text{def}}{=} \big(\{xyz\}, \varnothing\big)$ so that we have $\mathcal{I}(shw_1) =_\rho \mathcal{I}(shw_2)$ but $\varrho(shw_1) \neq \varrho(shw_2)$. While sharing between y and z is not contemplated in $\mathrm{rel}^{\mathrm{W}}\big(\{x\}, shw_1\big) = \big(\varnothing, \{x, xy, xz\}\big)$, the same does not hold for $\mathrm{rel}^{\mathrm{W}}\big(\{x\}, shw_2\big) = shw_2$.

A useful cautious widenings is the *gentle widening*, defined as follows. Consider $shw \in SH^{\mathrm{W}}$, and G be the graph (N, E) where $N \stackrel{\text{def}}{=} \big\{\, x \;\big|\; \{x\} \in \mathcal{I}(shw) \,\big\}$ and $E \stackrel{\text{def}}{=} \big\{\, (x, y) \;\big|\; \{x, y\} \in \mathcal{I}(shw), x, y \in N, x \neq y \,\big\}$. Then

$$\nabla^G\, shw \stackrel{\text{def}}{=} \big(\{C_1, \ldots, C_k\}, sh\big),$$

where $C_1, \ldots, C_k$ are all the maximal cliques of G. Note that, although the problem of enumerating all the maximal cliques of an undirected graph is NP-complete, this does not seem to be a problem for the graphs arising during the analysis of even the biggest real programs. For the experimentation we used the algorithm by Bron and Kerbosch [5], which is *Algorithm 457* in the ACM collection, even though more efficient algorithms are present in the literature.

Of intermediate precision is the widening based on Fecht's idea, which we call *Fecht's widening* and defined as

$$\nabla^F(cl, sh) \stackrel{\text{def}}{=} (cl \cup sh, \varnothing).$$

This widening is not cautious. However, it does not introduce new pairs. As it can introduce new singletons, it may destroy ground dependencies, and this is why this kind of widening is better coupled with *Pos*.

5 Experimental Evaluation

For the experimental evaluation of the Fecht's widening ∇^F, precision is compared with respect to the non-redundant sharing domain SH^ρ. In fact, this approach is almost always as precise as the optimal one using SH^ρ.

For this and the following experiments, the widening was guarded by a size threshold of 100 on the second component (i.e., the normal sharing part). In other words, immediately before each abstract mgu operation the analyzer operated ρ-redundancy elimination, as usual. If after this the operand (cl, sh) was such that $\#\, sh > 100$, then (cl, sh) was substituted by $\nabla^F(cl, sh)$. Let us call this guarded widening ∇^F_{100}. The results are reported in Table 2. Note that only the programs where the analysis with SH^{W} gives different results from the analysis with SH^ρ are reported in the table. Thus, for all the programs in the test-suite, the analysis with SH^{W} using the

	Goal-Independent						Goal-Dependent					
	P+WSH+F+L			P+NSH+F+L			P+WSH+F+L			P+NSH+F+L		
Program	NSP	GV	LV	NSP	GV	LV	NSP	GV	LV	NSP	GV	LV
`aqua_c`	11147	⋆406	2757	?	?	?	16364	⋆1188	2028	?	?	?
`caslog`	6553	474	1615	7027	474	1615	11338	1739	1062	?	?	?
`ochina`	2193	285	1166	2193	285	1166	3958	802	760	?	?	?
`quot_an`	288	37	160	288	37	160	639	159	122	646	159	122

Table 2: $Pos + SH^{\mathrm{W}} + Free + Lin$ vs $Pos + SH^{\rho} + Free + Lin$ using ∇^F_{100}: precision.

(rather drastic) widening ∇^F_{100} gives the same results obtainable (at a much higher cost) with SH^{ρ}, apart from those in Table 2. For `aqua_c` we obtain termination in reasonable time, as with Fecht's technique but with higher precision. The same holds for the GD analysis of `caslog` and `ochina`. However, while the GI analysis of `ochina` is "optimal" (meaning "as precise as SH^{ρ}"), this is not the case for `caslog`. Non-optimality happens also for the GD analysis of `quot_an`.

Obviously, ∇^F_{100} is never less precise than Fecht's domain. What is surprising, however, is that it is almost that efficient. The timings and the number of applications of the widening are reported in Table 3 for all the programs such that at least one timing was above 0.4 seconds. The first observation to be made is that the widening comes into play only a few times on the test-suite. On average, it is safe to say that on 99.9% of cases the sharing-sets remain of reasonable size (100 groups or less in this experiment). Table 3 says that this definition of "reasonable" makes sense: for those programs where widening does not take place the difference in performance between Fecht's domain and our SH^{W} with the ∇^F_{100} widening is very limited. Analysis of `aqua_c` shows that limiting precision may on occasions increase the cost (since less precision means more self-bin-unions to perform, thus even less precision, ...).

The results on the precision of ∇^F_{100} are so good that we are left with a ridiculous test-suite for checking how much we can improve by using a more cautious widening. Our experimentation showed that the gentle widening ∇^G_{100} improves over ∇^F_{100} only on `quot_an`. The same does, but at a lower price, a bigger widening ∇^g that is defined as ∇^G apart from the fact that singletons are disregarded. In other words, the undirected graph considered for ∇^g, given $shw \in SH^{\mathrm{W}}$, is $G \stackrel{\mathrm{def}}{=} (N, E)$ such that $E \stackrel{\mathrm{def}}{=} \{\, (x,y) \mid \{x,y\} \in \mathcal{I}(shw), x \neq y \,\}$ and $N \stackrel{\mathrm{def}}{=} \{\, x \mid (x,y) \in E \text{ or } (y,x) \in E \,\}$.

Now, suppose we perform sharing analysis *without* combining the sharing domain with *Pos*. Then it is important that the widening, as is the case with cautious ones, keeps track of singletons. Table 4 reports the results (fixpoint time and number of definitely not-sharing pairs) for SH^{W} with ∇^F_{100}, SH^{W} with ∇^g_{100}, and plain SH^{ρ}. The GD analysis of `bryant` is particularly eloquent example of the superiority of cautious widenings when *Pos* is not used.

	Goal-Independent			Goal-Dependent		
	P+DSH+F+L	P+WSH+F+L		P+DSH+F+L	P+WSH+F+L	
Program	T	T	#W	T	T	#W
action	0.1	0.1	1	1.1	1.4	1
aircraft	0.2	0.2	0	0.7	0.7	0
aqua_c	10.4	10.9	56	48.6	40.7	3
bmtp	0.8	0.9	6			
bryant	0.1	0.1	0	0.6	1.4	1
caslog	2.0	2.5	17	17.7	19.2	22
chat80	0.9	1.0	2	4.3	4.9	6
chat_parser	0.4	0.4	1	1.7	1.8	1
dpos_an	0.2	0.2	0	0.5	0.8	1
eliza	0.1	0.1	0	0.2	0.4	1
lg_sys	3.3	3.9	23			
log_interp	0.2	0.4	2	0.7	0.9	1
mixtus	0.9	0.9	4			
ochina	1.2	1.4	11	7.7	8.3	4
parser_dcg	0.2	0.1	0	0.7	0.6	0
peephole1	0.1	0.1	0	0.4	0.7	1
pets_an	0.8	0.9	4	4.5	4.5	1
peval	0.2	0.3	3	0.4	0.5	1
plaiclp	0.7	0.7	3			
press	0.1	0.1	0	0.4	0.7	0
quot_an	0.3	0.4	0	1.3	1.7	1
read	0.1	0.1	0	0.3	0.6	1
reg	0.4	0.4	4	0.4	0.4	1
sdda	0.1	0.1	1	0.2	0.4	2
sim	0.2	0.3	2	0.7	0.8	2
simple	0.1	0.2	0	0.6	0.9	2
slice	0.6	0.7	2			
spsys	0.6	0.7	5			
trs	0.2	0.3	2	0.5	0.6	1
unify	0.1	0.1	0	0.5	0.7	0

Table 3: $Pos + {\downarrow} SH + Free + Lin$ vs $Pos + SH^{\mathrm{W}} + Free + Lin$ using ∇^{F}_{100}: timings (T) and number of (sharing) widenings (#W).

6 Conclusion

We believe we have made a significant step forward towards the solution of the problem of practical, precise, and efficient sharing analysis of (constraint) logic programs. We have studied a new representation for set-sharing that allows for the incorporation of a variety of widenings. Extensive experimentation has shown that one of these widenings, which is based on an idea of C. Fecht, provides seemingly hard to beat precision and performance, when combined with *Pos*. When this combination is not performed, we have also shown that "more cautious" widenings offer more precision at an acceptable extra-cost.

	Goal-Independent						Goal-Dependent					
	$SH^{\mathrm{W}}, \nabla^{F}_{100}$		$SH^{\mathrm{W}}, \nabla^{g}_{100}$		SH^{ρ}		$SH^{\mathrm{W}}, \nabla^{F}_{100}$		$SH^{\mathrm{W}}, \nabla^{g}_{100}$		SH^{ρ}	
Program	T	NSP	T	NSP	T	NSP	T	NSP	T	NSP	T	NSP
aqua_c	4.1	10703	15.0	10899	?	?	27.6	15754	37.8	15754	?	?
bryant	0.2	1066	0.2	1066	0.2	1066	1.0	1033	1.0	1781	0.9	1781
caslog	2.1	6506	4.5	6539	744.4	7027	17.9	11054	21.8	11054	?	?
chat80	0.6	2536	1.1	2536	9.1	2536	4.4	3923	7.7	3926	285.7	5111
eliza	0.1	49	0.1	49	0.1	49	0.3	109	0.5	113	0.5	113
lg_sys	2.7	7328	7.9	7334	725.1	7334						
ochina	0.9	2187	2.6	2189	5.0	2193	5.9	3936	9.6	3936	?	?
pets_an	0.6	2525	1.3	2563	19.8	2569	3.7	4664	5.3	4664	1006.5	4710
quot_an	0.3	288	0.4	288	0.3	288	1.4	639	3.2	646	3.1	646
simple	0.1	373	0.1	373	0.1	373	0.8	572	1.3	639	17.6	639
slice	0.6	426	0.9	428	0.8	428						

Table 4: SH^{W} with ∇^{F}_{100} vs SH^{W} with ∇^{g}_{100} vs plain SH^{ρ}: timings and precision.

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