Exact Join Detection for Convex Polyhedra and Other Numerical Abstractions☆

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Abstract

Deciding whether the union of two convex polyhedra is a convex polyhedron is a basic problem in polyhedral computation having important applications in the field of constrained control and in the synthesis, analysis, verification and optimization of hardware and software systems. In these application fields, though, general convex polyhedra are only one among many so-called *numerical abstractions*: these range from restricted families of (not necessarily closed) convex polyhedra to non-convex geometrical objects. We thus tackle the problem from an abstract point of view: for a wide range of numerical abstractions that can be modeled as bounded join-semilattices —that is, partial orders where any finite set of elements has a least upper bound—, we show necessary and sufficient conditions for the equivalence between the lattice-theoretic join and the set-theoretic union. For the case of closed convex polyhedra —which, as far as we know, is the only one already studied in the literature— we improve upon the state-of-the-art by providing a new algorithm with a better worst-case complexity. The results and algorithms presented for the other numerical abstractions are new to this paper. All the algorithms have been implemented, experimentally validated, and made available in the Parma Polyhedra Library.

Key words: polyhedron, union, convexity, abstract interpretation, numerical abstraction, powerset domain.

1. Introduction

For $n \in \mathbb{N}$, let $\mathbb{D}_n \subset \wp(\mathbb{R}^n)$ be a set of finitely-representable sets such that $(\mathbb{D}_n, \subseteq)$ is a bounded join-semilattice, that is, a minimum element exists as well as the least upper bound for all $D_1, D_2 \in \mathbb{D}_n$. Such least upper bound —let us denote it by $D_1 \uplus D_2$ and call it the *join* of D_1 and D_2— is, of course, not guaranteed to be equal to $D_1 \cup D_2$. More generally, we refer to the problem of

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deciding, for each finite set $\{D_1, \dots, D_k\} \subseteq \mathbb{D}_n$, whether $\biguplus_{i=1}^k D_i = \bigcup_{i=1}^k D_i$ as the *exact join detection* problem.

Examples of $\mathbb{D}_n$ include n-dimensional convex polyhedra, either topologically closed or not necessarily so, restricted families of polyhedra characterized by interesting algorithmic complexities —such as *bounded-difference* and *octagonal shapes*—, Cartesian products of some families of intervals, and other "box-like" geometric objects where the intervals can have "holes" (for instance, Cartesian products of *modulo intervals* [33, 34] fall in this category). All these *numerical abstractions* allow to conveniently represent or approximate the constraints arising in constrained control (see, e.g., [26]) and, more generally, in the synthesis, analysis, verification and optimization of hardware and software systems (see, e.g., [8]).

The restrictions implied by convexity and/or by the "shapes" of the geometric objects in $\mathbb{D}_n$ are sometimes inappropriate for the application at hand. In these cases, one possibility is to consider finite sets of elements of $\mathbb{D}_n$. For instance, many applications in the field of hardware/software verification use constructions like the *finite powerset domain* of [1]: this is a special case of *disjunctive completion* [22], where disjunctions are implemented by maintaining an explicit (hence finite) and *non-redundant* collection of elements of $\mathbb{D}_n$. Non-redundancy means that a collection is made of maximal elements with respect to subset inclusion, so that no element is contained in another element in the collection. The finite powerset and similar constructions are such that $Q_1 = \{D_1, \dots, D_{h-1}, D_h, \dots, D_k\}$ and $Q_2 = \{D_1, \dots, D_{h-1}, D\}$ are two different representations for the same set, if $\bigcup_{i=h}^k D_i = \biguplus_{i=h}^k D_i = D$. The latter representation is clearly more desirable, and not only because —being more compact— it results in a better efficiency of all the involved algorithms. In the field of control engineering, the ability of efficiently simplifying Q_1 into Q_2 can be used to reduce the complexity of the solution to optimal control problems, thus allowing for the synthesis of cheaper control hardware [14, 37]. Similarly, the simplification of Q_1 into Q_2 can lead to improvements in loop optimizations obtained by automatic code generators such as CLooG [11]. In the same application area, this simplification allows for a reduction in the complexity of array data-flow analysis and for a simplification of *quasi-affine selection trees* (QUASTs). In loop optimization, dependences between program statements are modeled by parametric linear systems, whose solutions can be represented by QUASTs and computed by tools like PIP [23], which, however, can generate non-simplified QUASTs. These can be efficiently simplified provided there is an efficient procedure for deciding the exact join property. The exact join detection procedure can also be used as a preprocessing step for the extended convex hull problem [25].

Another important application of exact join detection comes from the field of static analysis via *abstract interpretation* [21, 22]. In abstract interpretation, static analysis is usually conducted by performing a fixpoint computation. Suppose we use the finite powerset domain $\bigl(\wp_{\mathrm{fn}}(\mathbb{D}_n), \sqsubseteq, \varnothing, \sqcup\bigr)$: this is the bounded join-semilattice of the *finite* and *non-redundant* subsets of $\mathbb{D}_n$ ordered by the relation given, for each $Q_1, Q_2 \in \wp_{\mathrm{fn}}(\mathbb{D}_n)$, by

$$Q_1 \sqsubseteq Q_2 \quad \iff \quad \forall D_1 \in Q_1 : \exists D_2 \in Q_2 \,.\, D_1 \subseteq D_2,$$

and '$\sqcup$' is the least upper bound (join) operator induced by '$\sqsubseteq$' [4]. The system under analysis is approximated by a monotonic (so called) *abstract semantic function* $\mathcal{A} \colon \wp_{\mathrm{fn}}(\mathbb{D}_n) \to \wp_{\mathrm{fn}}(\mathbb{D}_n)$, and the limit of the ascending chain given by $\mathcal{A}$'s iterates,

$$\mathcal{A}^0(\varnothing), \mathcal{A}^1(\varnothing), \mathcal{A}^2(\varnothing), \ldots, \tag{1}$$

is, by construction, a sound approximation of the analyzed system's behavior. Since $\wp_{\mathrm{fn}}(\mathbb{D}_n)$ has infinite ascending chains, the standard abstract iteration sequence (1) may converge very slowly or fail to converge altogether. For this reason, a *widening operator* $\nabla \colon \wp_{\mathrm{fn}}(\mathbb{D}_n)^2 \to \wp_{\mathrm{fn}}(\mathbb{D}_n)$ is introduced. This ensures that the sequence

$$\mathcal{B}^0(\varnothing), \mathcal{B}^1(\varnothing), \mathcal{B}^2(\varnothing), \ldots. \tag{2}$$

where, for each $Q \in \wp_{\mathrm{fn}}(\mathbb{D}_n)$, $\mathcal{B}(Q) := Q \mathbin{\nabla} \bigl(Q \sqcup \mathcal{A}(Q)\bigr)$, is ultimately stationary and that the (finitely computable) fixpoint of $\mathcal{B}$ is a post-fixpoint of $\mathcal{A}$, i.e., a sound approximation of the behavior of the system under consideration. In [4] three generic widening methodologies are presented for finite powerset abstract domains. A common trait of these methodologies is given by the fact that the precision/efficiency trade-off of the resulting widening can be greatly improved if domain elements are "pairwise merged" or even "fully merged." Let the cardinality of a finite set S be denoted by $\# S$. An element $Q = \{D_1, \ldots, D_h\}$ of $\wp_{\mathrm{fn}}(\mathbb{D}_n)$ is said to be *pairwise merged* if, for each $R \subseteq Q$, $\# R = 2$ implies $\bigcup R \neq \biguplus R$; the notion of being *fully merged* is obtained by replacing $\# R = 2$ with $\# R \geq 2$ in the above.

In this paper, we tackle the problem of exact join detection for all the numerical abstractions that are in widespread use at the time of writing.[1] This problem has been studied for convex polyhedra in [13]. We are not aware of any work that addresses the problem for other numerical abstractions.

In [13] the authors provide theoretical results and algorithms for the exact join detection problem applied to a pair of topologically closed convex polyhedra. Three different specializations of the problem are considered, depending on the chosen representation for the input polyhedra: H-polyhedra, described by constraints (halfspaces); V-polyhedra, described by generators (vertices); and VH-polyhedra, described by both constraints and generators.[2] The algorithms for the H and V representations, which are based on Linear Programming techniques, enjoy a polynomial worst-case complexity bound; the algorithm for VH-polyhedra achieves a better, strongly polynomial bound. For the H-polyhedra case only, it is also shown how the algorithm can be generalized to more than

[1] Since numerical abstractions are so critical in the field of hardware and software analysis and verification, new ones are proposed on a regular basis.

[2] The algorithms in [13] for the V and VH representations only consider the case of *bounded* polyhedra, i.e., polytopes; the extension to the unbounded case can be found in [12].

two input polyhedra. An improved theoretical result for the case of more than two V-polytopes is stated in [10].

The first contribution of the present paper is a theoretical result for the VH-polyhedra case, leading to the specification of a new algorithm improving upon the worst-case complexity bound of [12].

The second contribution is constituted by original results and algorithms concerning the exact join detection problem for the other numerical abstractions. For those that are restricted classes of topologically closed convex polyhedra, one could of course use the same algorithms used for the general case, but the efficiency would be poor. Consider that the applications of finite powersets of numerical abstractions range between two extremes:

- those using small-cardinality powersets of complex abstractions such as general polyhedra (see, for instance [16]);
- those using large-cardinality powersets of simple abstractions (for instance, verification tasks like the one described in [24], can be tackled this way).

So, in general, the simplicity of the abstractions is countered by their average number in the powersets. It is thus clear that specialized, efficient algorithms are needed for all numerical abstractions. In this paper we present algorithms, each backed with the corresponding correctness result, for the following numerical abstractions: not necessarily closed convex polyhedra, "box-like" geometric objects; rational (resp., integer) bounded difference shapes; and rational (resp., integer) octagonal shapes.

The plan of the paper is as follows. In Section 2 we introduce the required technical notation and terminology. In Section 3 we discuss the results and algorithms for convex polyhedra. The specialized results for boxes, bounded difference shapes and octagonal shapes are provided in Sections 4, 5 and 6, respectively. Section 7 concludes.

2. Preliminaries

The set of non-negative reals is denoted by $\mathbb{R}_+$. In the present paper, all topological arguments refer to the Euclidean topological space $\mathbb{R}^n$, for any positive integer n. If $S \subseteq \mathbb{R}^n$, then the *topological closure* of S is defined as $\mathbb{C}(S) := \bigcap \{ C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed} \}$.

For each $i \in \{1, \dots, n\}$, v_i denotes the i-th component of the (column) vector $\mathbf{v} \in \mathbb{R}^n$; the projection on space dimension i for a set $S \subseteq \mathbb{R}^n$ is denoted by $\pi_i(S) := \{ v_i \in \mathbb{R} \mid \mathbf{v} \in S \}$. We denote by $\mathbf{0}$ the vector of $\mathbb{R}^n$ having all components equal to zero. A vector $\mathbf{v} \in \mathbb{R}^n$ can also be interpreted as a matrix in $\mathbb{R}^{n \times 1}$ and manipulated accordingly with the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, which is denoted by $\mathbf{v}^{\mathrm{T}}$. The *scalar product* of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, denoted $\langle \mathbf{v}, \mathbf{w} \rangle$, is the real number $\mathbf{v}^{\mathrm{T}}\mathbf{w} = \sum_{i=1}^{n} v_i w_i$.

For any relational operator $\bowtie \in \{=, \leq, \geq, <, >\}$, we write $\mathbf{v} \bowtie \mathbf{w}$ to denote the conjunctive proposition $\bigwedge_{i=1}^{n} (v_i \bowtie w_i)$. Moreover, $\mathbf{v} \neq \mathbf{w}$ denotes the proposition $\neg(\mathbf{v} = \mathbf{w})$. We occasionally use the convenient notation $a \bowtie_1 b \bowtie_2 c$ to denote the conjunction $a \bowtie_1 b \wedge b \bowtie_2 c$ and do not distinguish conjunctions of propositions from sets of propositions.

2.1. Topologically Closed Convex Polyhedra

For each vector $\mathbf{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\mathbf{a} \neq \mathbf{0}$, the linear non-strict inequality constraint $\beta = \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \leq b\bigr)$ defines a topologically closed affine half-space of $\mathbb{R}^n$. The linear equality constraint $\langle \mathbf{a}, \mathbf{x} \rangle = b$ defines an affine hyperplane. A topologically closed convex polyhedron is usually described as a finite system of linear equality and non-strict inequality constraints. Theoretically speaking, it is simpler to express each equality constraint as the intersection of the two half-spaces $\langle \mathbf{a}, \mathbf{x} \rangle \leq b$ and $\langle -\mathbf{a}, \mathbf{x} \rangle \leq -b$. We do not distinguish between syntactically different constraints defining the same affine half-space so that, e.g., $x \leq 2$ and $2x \leq 4$ are considered to be the same constraint.

We write $\mathrm{con}(\mathcal{C})$ to denote the polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ described by the finite *constraint system* $\mathcal{C}$. Formally, we define

$$\mathrm{con}(\mathcal{C}) := \left\{ \mathbf{p} \in \mathbb{R}^n \;\middle|\; \forall \beta = \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \leq b\bigr) \in \mathcal{C} : \langle \mathbf{a}, \mathbf{p} \rangle \leq b \right\}.$$

The function 'con' enjoys an anti-monotonicity property, meaning that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ implies $\mathrm{con}(\mathcal{C}_1) \supseteq \mathrm{con}(\mathcal{C}_2)$.

Alternatively, the definition of a topologically closed convex polyhedron can be based on some of its geometric features. A vector $\mathbf{r} \in \mathbb{R}^n$ such that $\mathbf{r} \neq \mathbf{0}$ is a *ray* (or *direction of infinity*) of a non-empty polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ if, for every point $\mathbf{p} \in \mathcal{P}$ and every non-negative scalar $\rho \in \mathbb{R}_+$, we have $\mathbf{p} + \rho\mathbf{r} \in \mathcal{P}$; the set of all the rays of a polyhedron $\mathcal{P}$ is denoted by $\mathrm{rays}(\mathcal{P})$. A vector $\mathbf{l} \in \mathbb{R}^n$ is a *line* of $\mathcal{P}$ if both $\mathbf{l}$ and $-\mathbf{l}$ are rays of $\mathcal{P}$. The empty polyhedron has no rays and no lines. As was the case for equality constraints, the theory can dispense with the use of lines by using the corresponding pair of rays. Moreover, when vectors are used to denote rays, no distinction is made between different vectors having the same direction so that, e.g., $\mathbf{r}_1 = (1, 3)^{\mathrm{T}}$ and $\mathbf{r}_2 = (2, 6)^{\mathrm{T}}$ are considered to be the same ray in $\mathbb{R}^2$. The following theorem is a simple consequence of well-known theorems by Minkowski and Weyl [36].

Theorem 2.1. *The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a closed polyhedron if and only if there exist finite sets $R, P \subseteq \mathbb{R}^n$ of cardinality r and p, respectively, such that $\mathbf{0} \notin R$ and*

$$\mathcal{P} = \mathrm{gen}\bigl((R, P)\bigr) := \left\{ R\rho + P\sigma \in \mathbb{R}^n \;\middle|\; \rho \in \mathbb{R}^r_+, \sigma \in \mathbb{R}^p_+, \sum_{i=1}^{p} \sigma_i = 1 \right\}.$$

When $\mathcal{P} \neq \varnothing$, we say that $\mathcal{P}$ is described by the *generator system* $\mathcal{G} = (R, P)$. In particular, the vectors of R and P are rays and points of $\mathcal{P}$, respectively. Thus, each point of the generated polyhedron is obtained by adding a non-negative combination of the rays in R and a convex combination of the points in P. Informally speaking, if no "supporting point" is provided then an empty polyhedron is obtained; formally, $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. By convention, the empty system (i.e., the system with $R = \varnothing$ and $P = \varnothing$) is the only generator system for the empty polyhedron. We define a partial order relation '$\sqsubseteq$' on generator systems, which is the component-wise extension of set inclusion. Namely, for any generator systems $\mathcal{G}_1 = (R_1, P_1)$ and $\mathcal{G}_2 = (R_2, P_2)$, we have $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ if and only if $R_1 \subseteq R_2$ and $P_1 \subseteq P_2$; if, in addition, $\mathcal{G}_1 \neq \mathcal{G}_2$, we write $\mathcal{G}_1 \sqsubset \mathcal{G}_2$. The function 'gen' enjoys a monotonicity property, as $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ implies $\mathrm{gen}(\mathcal{G}_1) \subseteq \mathrm{gen}(\mathcal{G}_2)$.

The vector $\mathbf{v} \in \mathcal{P}$ is an *extreme point* (or *vertex*) of the polyhedron $\mathcal{P}$ if it cannot be expressed as a convex combination of some other points of $\mathcal{P}$. Similarly, $\mathbf{r} \in \mathrm{rays}(\mathcal{P})$ is an *extreme ray* of $\mathcal{P}$ if it cannot be expressed as a non-negative combination of some other rays of $\mathcal{P}$. It is worth stressing that, in general, the vectors in R and P are not the extreme rays and the vertices of the polyhedron: for instance, any half-space of $\mathbb{R}^2$ has two extreme rays and no vertices, but any generator system describing it will contain at least three rays and one point.

The combination of the two approaches outlined above is the basis of the double description method due to Motzkin et al. [32], which exploits the duality principle to compute each representation starting from the other one, possibly minimizing both descriptions. Clever implementations of this *conversion* procedure, such as those based on the extension by Le Verge [29] of Chernikova's algorithms [17, 18, 19], are the starting point for the development of software libraries based on the double description method. While being characterized by a worst-case computational cost that is exponential in the size of the input, these algorithms turn out to be practically useful for the purposes of many applications in the context of static analysis.

We denote by $\mathbb{CP}_n$ the set of all topologically closed polyhedra in $\mathbb{R}^n$, which is partially ordered by subset inclusion to form a non-complete lattice; the finitary greatest lower bound operator corresponds to intersection; the finitary least upper bound operator, denoted by '$\uplus$', corresponds to the convex polyhedral hull.

2.2. Not Necessarily Closed Convex Polyhedra

The linear strict inequality constraint $\beta = \bigl(\langle \mathbf{a}, \mathbf{x} \rangle > b\bigr)$ defines a topologically open affine half-space of $\mathbb{R}^n$. A not necessarily closed (NNC) convex polyhedron is defined by a finite system of strict and non-strict inequality constraints. Since by using lines, rays and points we can only represent topologically closed polyhedra, the key step for a parametric description of NNC polyhedra is the introduction of a new kind of generator called a *closure point* [3].

Definition 2.2. (Closure point.) A vector $\mathbf{c} \in \mathbb{R}^n$ is a *closure point* of $S \subseteq \mathbb{R}^n$ if and only if $\mathbf{c} \in \mathbb{C}(S)$.

For a non-empty NNC polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, a vector $\mathbf{c} \in \mathbb{R}^n$ is a closure point of $\mathcal{P}$ if and only if $\sigma\mathbf{p} + (1 - \sigma)\mathbf{c} \in \mathcal{P}$ for every point $\mathbf{p} \in \mathcal{P}$ and every $\sigma \in \mathbb{R}$ such that $0 < \sigma < 1$. By excluding the case when $\sigma = 0$, $\mathbf{c}$ is not forced to be in $\mathcal{P}$.

The following theorem taken from [3] is a generalisation of Theorem 2.1 to NNC polyhedra.

Theorem 2.3. *The set $\mathcal{P} \subseteq \mathbb{R}^n$ is an NNC polyhedron if and only if there exist finite sets $R, P, C \subseteq \mathbb{R}^n$ of cardinality r, p and c, respectively, such that $\mathbf{0} \notin R$ and*

$$\mathcal{P} = \operatorname{gen}\bigl((R, P, C)\bigr) := \left\{ R\rho + P\sigma + C\tau \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \rho \in \mathbb{R}^r_+, \sigma \in \mathbb{R}^p_+, \sigma \neq \mathbf{0}, \\ \tau \in \mathbb{R}^c_+, \\ \sum_{i=1}^p \sigma_i + \sum_{i=1}^c \tau_i = 1 \end{array} \right\}.$$

When $\mathcal{P} \neq \varnothing$, we say that $\mathcal{P}$ is described by the *extended generator system* $\mathcal{G} = (R, P, C)$. As was the case for closed polyhedra, the vectors in R and P are rays and points of $\mathcal{P}$, respectively. The condition $\sigma \neq \mathbf{0}$ ensures that at least one of the points of P plays an active role in any convex combination of the vectors of P and C. The vectors of C are closure points of $\mathcal{P}$. Since both rays and closure points need a supporting point, we have $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. The partial order relation '$\sqsubseteq$' on generator systems is easily extended to also take into account the closure points component, so that the overloading of the function 'gen' still satisfies the monotonicity property.

The set of all NNC polyhedra in $\mathbb{R}^n$, denoted $\mathbb{P}_n$, is again a non-complete lattice partially ordered by subset inclusion, having $\mathbb{CP}_n$ as a sublattice.

3. Exact Join Detection for Convex Polyhedra

In this section, we provide results for the exact join detection problem for convex polyhedra. Here we just consider the case when a double description representation is available; that is, in the proposed methods, we exploit both the constraint and the generator descriptions of the polyhedra.

3.1. Topologically Closed Polyhedra

First we consider the exact join detection problem for closed polyhedra since, in this case, given any two closed polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{CP}_n$, we have that $\mathcal{P}_1 \cup \mathcal{P}_2$ is convex if and only if $\mathcal{P}_1 \uplus \mathcal{P}_2 = \mathcal{P}_1 \cup \mathcal{P}_2$. Before stating and proving the main result for this section, we present the following lemma that establishes some simple conditions that will ensure the union of two closed polyhedra is not convex.

Lemma 3.1. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{CP}_n$ be topologically closed non-empty polyhedra. Suppose there exist a constraint β and a vector $\mathbf{p}$ such that* (1) *$\mathbf{p}$ saturates β,* (2) *β is satisfied by $\mathcal{P}_1$ but violated by $\mathcal{P}_2$, and* (3) *$\mathbf{p} \in \mathcal{P}_1 \setminus \mathcal{P}_2$. Then, $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex.*

Proof. By (2), there exists a point $\mathbf{p}_2 \in \mathcal{P}_2$ that violates β. Consider the closed line segment $s := [\mathbf{p}, \mathbf{p}_2]$; by (1), the one and only point on this segment that satisfies β is $\mathbf{p}$; by (3), $\mathbf{p} \in \mathcal{P}_1$ so that $s \subseteq \mathcal{P}_1 \uplus \mathcal{P}_2$. Also by (3), $\mathbf{p} \notin \mathcal{P}_2$, so that there exists a non-strict constraint β_2 that is satisfied by $\mathcal{P}_2$ but violated by $\mathbf{p}$. Since $\mathbf{p}_2 \in \mathcal{P}_2$, there exists a vector $\mathbf{q} \in s$ that saturates β_2 and $\mathbf{q} \neq \mathbf{p}$. It follows that the open line segment $s_1 := (\mathbf{p}, \mathbf{q})$ is non-empty and every point in s_1 violates both β and β_2; hence $s_1 \cap \mathcal{P}_1 = s_1 \cap \mathcal{P}_2 = \varnothing$. However, by construction,

$$(\mathbf{p}, \mathbf{q}) \subset [\mathbf{p}, \mathbf{p}_2] \subseteq \mathcal{P}_1 \uplus \mathcal{P}_2,$$

so that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. Therefore $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex. □

Theorem 3.2. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{CP}_n$ be topologically closed non-empty polyhedra, where $\mathcal{P}_1 = \mathrm{con}(\mathcal{C}_1) = \mathrm{gen}(\mathcal{G}_1)$. Then $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$ if and only if there exist a constraint $\beta_1 \in \mathcal{C}_1$ and a generator g_1 in $\mathcal{G}_1$ such that* (1) *g_1 saturates β_1,* (2) *$\mathcal{P}_2$ violates β_1, and* (3) *$\mathcal{P}_2$ does not subsume g_1.*

Proof. Suppose first that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex and there exist points $\mathbf{p}_1 \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $\mathbf{p}_2 \in \mathcal{P}_2 \setminus \mathcal{P}_1$ such that $[\mathbf{p}_1, \mathbf{p}_2] \nsubseteq \mathcal{P}_1 \cup \mathcal{P}_2$. As $\mathbf{p}_2 \notin \mathcal{P}_1$, there exists a constraint $\beta_1 \in \mathcal{C}_1$ such that $\mathbf{p}_2$ violates β_1; also, there exists a point $\mathbf{p}' \in [\mathbf{p}_1, \mathbf{p}_2]$ that saturates β_1; moreover $\mathbf{p}' \notin \mathcal{P}_2$ since if this held, we would have $[\mathbf{p}_1, \mathbf{p}'] \subseteq \mathcal{P}_1$ and $[\mathbf{p}', \mathbf{p}_2] \subseteq \mathcal{P}_2$, contradicting $[\mathbf{p}_1, \mathbf{p}_2] \nsubseteq \mathcal{P}_1 \cup \mathcal{P}_2$. Let $\mathcal{G}_1'$ be the generator system containing all the points and rays in $\mathcal{G}_1$ that saturate β_1. Then $\mathbf{p}' \in \mathrm{gen}(\mathcal{G}_1')$. Hence, as $\mathbf{p}' \notin \mathcal{P}_2$, there is a point or ray g_1 in $\mathcal{G}_1'$ that is not subsumed by $\mathcal{P}_2$. Hence conditions (1), (2) and (3) are all satisfied.

Suppose now that there exist a constraint $\beta_1 \in \mathcal{C}_1$ and a generator g_1 in $\mathcal{G}_1$ such that conditions (1), (2) and (3) hold. Then, as $\mathcal{P}_1 = \mathrm{gen}(g_1)$, β_1 is satisfied by $\mathcal{P}_1$. If $g_1 := \mathbf{p}_1$ is a point, then, by letting $\beta := \beta_1$ and $\mathbf{p} := \mathbf{p}_1$ in Lemma 3.1, the required three conditions hold so that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex. Now suppose that $g_1 := \mathbf{r}_1$ is a ray for $\mathcal{P}_1$. Suppose there exists a point $\mathbf{p}_1' \in \mathcal{P}_1$ that saturates the constraint β_1. By condition (3), $\mathbf{r}_1$ is not a ray for $\mathcal{P}_2$; hence for some $\rho \in \mathbb{R}_+$ there exists a point $\mathbf{p}_1 := \mathbf{p}_1' + \rho \mathbf{r}_1 \in \mathcal{P}_1 \setminus \mathcal{P}_2$ that also saturates β_1. Hence, letting $\beta := \beta_1$ and $\mathbf{p} := \mathbf{p}_1$ in Lemma 3.1, the required three conditions hold so that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex. Otherwise, no point in $\mathcal{P}_1$ saturates β_1.[3] Suppose, for some $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, $\beta = \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \bowtie b\bigr)$; then, since $\mathcal{P}_1 \neq \varnothing$, there exist a point $\mathbf{p}_1' \in \mathcal{P}_1$ and a constraint $\beta_1' := \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \bowtie b'\bigr)$ such that $\mathcal{P}_1$ satisfies β_1' and $\mathbf{p}_1'$ saturates β_1'; note that β_1' is also saturated by ray $\mathbf{r}_1$. Thus we can construct, as done above, a point $\mathbf{p}_1 := \mathbf{p}_1' + \rho \mathbf{r}_1 \in \mathcal{P}_1 \setminus \mathcal{P}_2$ that saturates β_1'. Hence, letting $\beta := \beta_1$ and $\mathbf{p} := \mathbf{p}_1$ in Lemma 3.1, the required three conditions hold so that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex. Therefore, in all cases, $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. □

[3] This may happen because we made no minimality assumption on the constraint system $\mathcal{C}_1$, so that β_1 may be redundant.

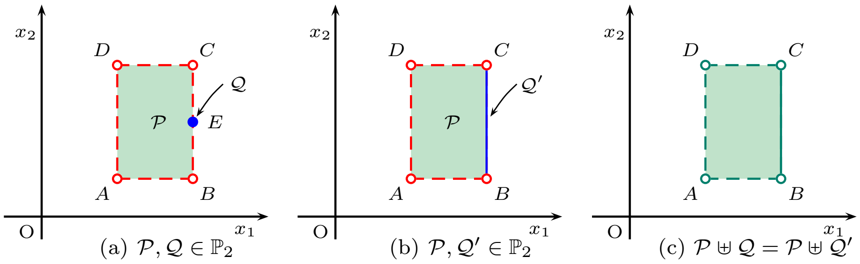


Figure 1: The convex polyhedral hull of NNC polyhedra

Example 3.3. Consider the closed polyhedra

$$\begin{aligned}\mathcal{P}_1 &= \mathrm{con}(\mathcal{C}_1) = \mathrm{con}\bigl(\{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 2\}\bigr)\\ &= \mathrm{gen}(\mathcal{G}_1) = \mathrm{gen}\bigl((\varnothing, P)\bigr),\\ \mathcal{P}_2 &= \mathrm{con}(\mathcal{C}_2) = \mathrm{con}\bigl(\{x_1 \leq 2, x_2 \geq 0, x_1 - x_2 \geq 0\}\bigr),\end{aligned}$$

where $P = \bigl\{(0,0)^{\mathrm{T}}, (2,0)^{\mathrm{T}}, (0,2)^{\mathrm{T}}\bigr\}$. Then

$$\mathcal{P}_1 \uplus \mathcal{P}_2 = \mathrm{con}\bigl(\{x_1 \geq 0, x_2 \geq 0, x_1 \leq 2, x_2 \leq 2\}\bigr)$$

so that $(1,1)^{\mathrm{T}} \in (\mathcal{P}_1 \uplus \mathcal{P}_2) \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$ and, hence, $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. In Theorem 3.2, let $\beta_1 = (x_1 + x_2 \leq 2)$ and $g_1 = (0,2)^{\mathrm{T}}$. Then conditions (1), (2) and (3) are all satisfied.

For each $i \in \{1,2\}$, let l_i and m_i denote the number of constraints in $\mathcal{C}_i$ and generators in $\mathcal{G}_i$, respectively. Then, the worst-case complexity of an algorithm based on Theorem 3.2, computed by summing the complexities for checking each of the conditions (1), (2) and (3), is in $\mathrm{O}\bigl(n(l_1 m_1 + l_1 m_2 + l_2 m_1)\bigr)$. Notice that the complexity bound is not symmetric so that, if $l_1 m_1 \gg l_2 m_2$ holds, then an efficiency improvement can be obtained by exchanging the roles of $\mathcal{P}_1$ and $\mathcal{P}_2$ in the theorem. In all cases, an improvement is obtained with respect to the $\mathrm{O}\bigl(n(l_1 + l_2) m_1 m_2\bigr)$ complexity bound of Algorithm 7.1 in [13].

3.2. Not Necessarily Closed Polyhedra

We now consider the exact join detection problem for two NNC polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$; in this case, it can happen that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$ although the union $\mathcal{P}_1 \cup \mathcal{P}_2$ is convex.

Example 3.4. Consider the NNC polyhedra $\mathcal{P}$ and $\mathcal{Q}$ in Figure 1(a), where $\mathcal{P}$ is the open rectangle $ABCD$ and $\mathcal{Q}$ is the single point E. The union $\mathcal{P} \cup \mathcal{Q}$ is convex but it is not an NNC polyhedron: the convex polyhedral hull $\mathcal{P} \uplus \mathcal{Q}$ (see Figure 1(c)) also contains the line segment (B, C) and hence $\mathcal{P} \uplus \mathcal{Q} \neq \mathcal{P} \cup \mathcal{Q}$. On the other hand, if we now consider $\mathcal{P}$ and $\mathcal{Q}'$, as shown in Figure 1(b), where $\mathcal{Q}'$ is the line segment (B, C), then the convex polyhedral hull $\mathcal{P} \uplus \mathcal{Q}'$ is such that $\mathcal{P} \uplus \mathcal{Q}' = \mathcal{P} \uplus \mathcal{Q} = \mathcal{P} \cup \mathcal{Q}'$.

Before stating and proving the main result for this section, we present a lemma similar to Lemma 3.1 but generalized so as to apply to NNC polyhedra.

Lemma 3.5. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$ be non-empty polyhedra. Suppose that there exist a constraint β and a vector $\mathbf{p}$ such that* (1) *$\mathbf{p}$ saturates β,* (2) *β is satisfied by $\mathcal{P}_1$ but violated by $\mathcal{P}_2$, and* (3) *$\mathbf{p} \in \mathbb{C}(\mathcal{P}_1) \setminus \mathbb{C}(\mathcal{P}_2)$. Then $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$.*

Proof. By (2), there exists a point $\mathbf{p}_2 \in \mathcal{P}_2$ that violates β. Consider the line segment $s := (\mathbf{p}, \mathbf{p}_2]$; by (1), no point on s satisfies β; by (3), $\mathbf{p} \in \mathbb{C}(\mathcal{P}_1)$ so that $s \subseteq \mathcal{P}_1 \uplus \mathcal{P}_2$. Also, by (3), $\mathbf{p} \notin \mathbb{C}(\mathcal{P}_2)$, so that, there exists a constraint β_2 that is satisfied by $\mathbb{C}(\mathcal{P}_2)$ but violated by $\mathbf{p}$. Since $\mathbf{p} \notin \mathcal{P}_2$ and $\mathbf{p}_2 \in \mathcal{P}_2$, there exists a vector $\mathbf{q} \in s$ that saturates β_2 and $\mathbf{q} \neq \mathbf{p}$. It follows that the open line segment $s_1 := (\mathbf{p}, \mathbf{q})$ is non-empty and every point in s_1 violates both β and β_2; hence $s_1 \cap \mathcal{P}_1 = s_1 \cap \mathcal{P}_2 = \varnothing$. However, by construction,

$$(\mathbf{p}, \mathbf{q}) \subset (\mathbf{p}, \mathbf{p}_2] \subseteq (\mathcal{P}_1 \uplus \mathcal{P}_2),$$

so that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. $\square$

Theorem 3.6. *For $i \in \{1, 2\}$, let $\mathcal{P}_i = \mathrm{con}(\mathcal{C}_i) = \mathrm{gen}(\mathcal{G}_i) \in \mathbb{P}_n$ be non-empty polyhedra. Then $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$ if and only if, for some $i, j \in \{1, 2\}$, $i \neq j$, there exists a generator g_i in $\mathcal{G}_i$ that saturates a constraint $\beta_i \in \mathcal{C}_i$ violated by $\mathcal{P}_j$ and at least one of the following hold:*

(1) *g_i is a ray or closure point in $\mathcal{G}_i$ that is not subsumed by $\mathcal{P}_j$;*

(2) *g_i is a point in $\mathcal{G}_i$, β_i is non-strict and $g_i \notin \mathbb{C}(\mathcal{P}_j)$;*

(3) *g_i is a closure point in $\mathcal{G}_i$, β_i is strict and $g_i \in (\mathcal{P}_1 \uplus \mathcal{P}_2) \setminus \mathcal{P}_j$.*

Proof. Suppose first that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. Then, for some $i, j \in \{1, 2\}$, $i \neq j$, there exist points $\mathbf{p}_i \in \mathbb{C}(\mathcal{P}_i) \setminus \mathcal{P}_j$ and $\mathbf{p}_j \in \mathcal{P}_j \setminus \mathcal{P}_i$ such that $(\mathbf{p}_i, \mathbf{p}_j] \nsubseteq \mathcal{P}_1 \cup \mathcal{P}_2$. By construction, $(\mathbf{p}_i, \mathbf{p}_j] \subseteq \mathcal{P}_1 \uplus \mathcal{P}_2$. For ease of notation, we will assume that $i = 1$ and $j = 2$; the other case follows by a symmetrical argument. As $\mathbf{p}_2 \notin \mathcal{P}_1$, there exists a constraint $\beta_1 \in \mathcal{C}_1$ and a point $\mathbf{p} \in [\mathbf{p}_1, \mathbf{p}_2]$ such that $\mathbf{p}_2$ violates β_1, $\mathbf{p}$ saturates β_1 and $\mathbf{p} \in \mathbb{C}(\mathcal{P}_1)$; moreover, if $\mathbf{p} \notin \mathcal{P}_1$, then we can and will assume that β_1 is a strict constraint. Note that $\mathbf{p} \notin \mathcal{P}_2$ since, if this held, we would have $(\mathbf{p}_1, \mathbf{p}) \subseteq \mathcal{P}_1$ and $[\mathbf{p}, \mathbf{p}_2] \subseteq \mathcal{P}_2$, contradicting $(\mathbf{p}_1, \mathbf{p}_2] \nsubseteq \mathcal{P}_1 \cup \mathcal{P}_2$. Moreover, if $\mathbf{p} \in \mathcal{P}_1$, $\mathbf{p} \notin \mathbb{C}(\mathcal{P}_2)$ since, if this held, we would have $(\mathbf{p}_1, \mathbf{p}) \subseteq \mathcal{P}_1$ and $[\mathbf{p}, \mathbf{p}_2] \subseteq \mathcal{P}_2$, again contradicting $(\mathbf{p}_1, \mathbf{p}_2] \nsubseteq \mathcal{P}_1 \cup \mathcal{P}_2$. Let $\mathcal{G}'_1 = (R'_1, P'_1, C'_1)$ be the system of all the generators in $\mathcal{G}_1$ that saturate β_1 so that $\mathbf{p} \in \mathrm{gen}\bigl((R'_1, P'_1 \cup C'_1, \varnothing)\bigr)$. Suppose condition (1) does not hold; that is, suppose that all the rays in R'_1 are subsumed by $\mathcal{P}_2$ and $C'_1 \subseteq \mathbb{C}(\mathcal{P}_2)$. If β_1 is non-strict, $\mathbf{p} \in \mathcal{P}_1$ so that $\mathbf{p} \notin \mathbb{C}(\mathcal{P}_2)$; hence $P'_1 \nsubseteq \mathbb{C}(\mathcal{P}_2)$ and condition (2) holds. Suppose now that β_1 is strict; then $P'_1 = \varnothing$, $C'_1 \neq \varnothing$ and $C'_1 \cap \mathcal{P}_1 = \varnothing$. Hence, as $\mathbf{p} \notin \mathcal{P}_2$, there exists $C''_1 \subseteq C'_1$ such that $\mathbf{p} \in \mathrm{gen}\bigl((R'_1, C''_1, \varnothing)\bigr)$ and $C''_1 \cap \mathcal{P}_2 = \varnothing$. As $\mathbf{p} \in \mathcal{P}_1 \uplus \mathcal{P}_2$, there exists $g_1 \in C''_1$ such that $g_1 \in \mathcal{P}_1 \uplus \mathcal{P}_2$; hence condition (3) holds.

Suppose now that, for some $i, j \in \{1, 2\}$ $i \neq j$, there exists a generator g_i in $\mathcal{G}_i$ that saturates a constraint $\beta_i \in \mathcal{C}_i$ that is violated by $\mathcal{P}_j$ and at least one of conditions (1), (2) or (3) holds. As before, we will assume that $i = 1$ and $j = 2$, since the other case follows by a symmetrical argument. Let $\beta_1 := \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \bowtie b\bigr)$, where $\bowtie \in \{<, \leq\}$. If condition (1) holds, then g_1 is a closure point or ray that is not subsumed by $\mathcal{P}_2$. Thus, if g_1 is a closure point in $\mathcal{G}_1$, $g_1 \notin \mathbb{C}(\mathcal{P}_2)$; so that, by letting $\beta := \beta_1$ and $\mathbf{p} := g_1$ in Lemma 3.5, $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. Suppose now that $g_1 := \mathbf{r}_1$ is a ray. Since $\mathcal{P}_1 \neq \varnothing$, there exist a point $\mathbf{p}'_1 \in \mathbb{C}(\mathcal{P}_1)$ and a constraint $\beta'_1 := \bigl(\langle \mathbf{a}, \mathbf{x} \rangle \leq \langle \mathbf{a}, \mathbf{p}'_1 \rangle\bigr)$ such that $\mathcal{P}_1$ satisfies β'_1; note that, by definition, β'_1 is saturated by the point $\mathbf{p}'_1$ and the ray $\mathbf{r}_1$.[4] Thus there exists $\rho \in \mathbb{R}_+$ such that the point $\mathbf{p}_1 := \mathbf{p}'_1 + \rho \mathbf{r}_1 \in \mathbb{C}(\mathcal{P}_1)$ does not belong to $\mathbb{C}(\mathcal{P}_2)$; hence, as $\mathbf{p}_1$ saturates β'_1, by letting $\beta := \beta'_1$ and $\mathbf{p} := \mathbf{p}_1$ in Lemma 3.5, $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. If condition (2) holds, then g_1 is a point in $\mathcal{G}_1$ (so that $g_1 \in \mathcal{P}_1$) and $g_1 \notin \mathbb{C}(\mathcal{P}_2)$. Then, by letting $\beta := \beta_1$ and $\mathbf{p} := g_1$ in Lemma 3.5, $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. Finally, if condition (3) holds, then g_1 is a closure point in $\mathcal{G}_1$ and β_1 is a strict constraint, so that $g_1 \notin \mathcal{P}_1$; as condition (3) also states that $g_1 \in (\mathcal{P}_1 \uplus \mathcal{P}_2) \setminus \mathcal{P}_2$, it follows that $\mathcal{P}_1 \uplus \mathcal{P}_2 \neq \mathcal{P}_1 \cup \mathcal{P}_2$. □

Observe that the conditions stated for the NNC case in Theorem 3.6 are more involved than the conditions stated for the topologically closed case in Theorem 3.2. In particular, a direct correspondence can only be found for condition (2) of Theorem 3.6. The added complexity is justified by the need to properly capture special cases where, as said above, convexity alone is not sufficient. Hence, we provide examples showing cases when conditions (1) and (3) of Theorem 3.6 come into play.

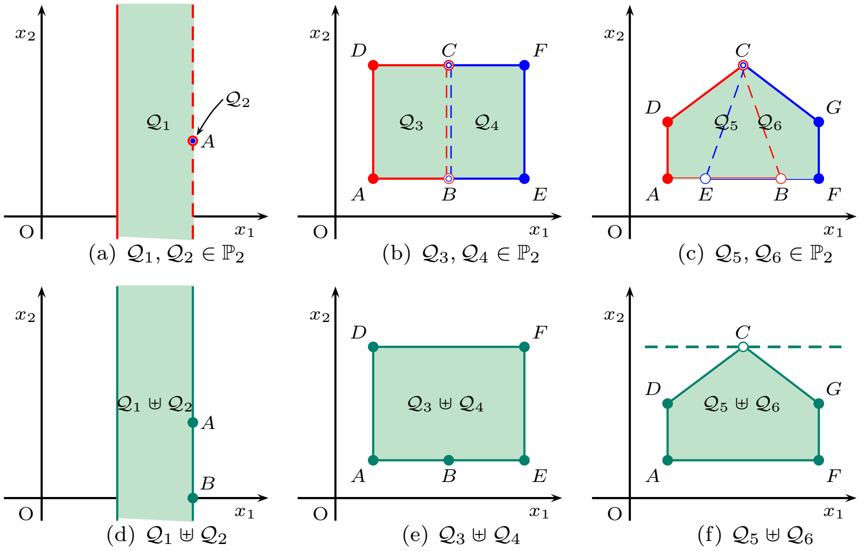
Example 3.7 (Condition (1) of Theorem 3.6). We first show how condition (1) of Theorem 3.6 where g_1 is a closure point can properly discriminate between the two cases illustrated in Figures 1(a) and 1(b).

Consider the polyhedra $\mathcal{P}$ and $\mathcal{Q}$ in Figure 1(a) and assume that the line segment (B, C) satisfies the constraint $x_1 = 4$. In the statement of Theorem 3.6, let $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2 = \mathcal{Q}$, $i = 1$, $j = 2$, $\beta_1 = (x_1 < 4) \in \mathcal{C}_1$ and $g_1 = B$ be a closure point in $\mathcal{G}_1$. Then β_1 is violated by $\mathcal{P}_2$ and saturated by g_1, but g_1 is not subsumed by $\mathcal{P}_2$. Hence condition (1) of Theorem 3.6 holds and we correctly conclude that $\mathcal{P} \uplus \mathcal{Q} \neq \mathcal{P} \cup \mathcal{Q}$.

On the other hand, if we consider polyhedra $\mathcal{P}$ and $\mathcal{Q}'$ in Figure 1(b) and let $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}_2 = \mathcal{Q}'$, then the closure point $g_1 = B$ is subsumed by $\mathcal{P}_2$ so that condition (1) of Theorem 3.6 does not hold.

Note that such a discrimination could not be obtained by checking only condition (2) of Theorem 3.6. If we swap the indices i and j so that $i = 2$, $j = 1$; letting $\beta_2 = (x_1 \geq 4) \in \mathcal{C}_2$ and $g_2 = E$ be a point in $\mathcal{G}_2$, then $g_2 \in \mathbb{C}(\mathcal{P})$ and β_2 is a non-strict constraint of both $\mathcal{Q}$ and $\mathcal{Q}'$ violated by $\mathcal{P}$ and saturated by point g_2; hence condition (2) does not hold for both $\mathcal{P}_2 = \mathcal{Q}$ and for $\mathcal{P}_2 = \mathcal{Q}'$.

[4] Note that $\langle \mathbf{a}, \mathbf{p}'_1 \rangle$ may differ from b because we made no minimality assumption on the constraint system $\mathcal{C}_1$, so that β_1 may be redundant.



(a) $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{P}_2$ (b) $\mathcal{Q}_3, \mathcal{Q}_4 \in \mathbb{P}_2$ (c) $\mathcal{Q}_5, \mathcal{Q}_6 \in \mathbb{P}_2$

(d) $\mathcal{Q}_1 \uplus \mathcal{Q}_2$ (e) $\mathcal{Q}_3 \uplus \mathcal{Q}_4$ (f) $\mathcal{Q}_5 \uplus \mathcal{Q}_6$

Figure 2: More examples for the convex polyhedral hull of NNC polyhedra

For an example of application of condition (1) of Theorem 3.6 when g_1 is a ray, consider $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in Figure 2(a), where $\mathcal{Q}_1 = \operatorname{con}\bigl(\{2 \leq x_1 < 4\}\bigr)$ is an unbounded strip and $\mathcal{Q}_2 = \{A\}$ is a singleton, with $A = (4,2)^{\mathrm{T}}$. It can be seen that $\mathcal{Q}_1 \uplus \mathcal{Q}_2$, the polyhedron in Figure 2(d), contains the point $B = (4,0)^{\mathrm{T}}$ which is not a point in $\mathcal{Q}_1$ or $\mathcal{Q}_2$, so that $\mathcal{Q}_1 \uplus \mathcal{Q}_2 \neq \mathcal{Q}_1 \cup \mathcal{Q}_2$. In the statement of Theorem 3.6, let $\mathcal{P}_1 = \mathcal{Q}_1$, $\mathcal{P}_2 = \mathcal{Q}_2$, $i = 1$, $j = 2$, $\beta_1 = (x_1 < 4) \in \mathcal{C}_1$ and $g_1 = (0,1)^{\mathrm{T}}$ be a ray in $\mathcal{G}_1$. Then β_1 is violated by $\mathcal{P}_2$ and saturated by the ray g_1; but g_1 is not subsumed by $\mathcal{P}_2$ so that condition (1) of Theorem 3.6 holds.

Example 3.8 (Condition (3) of Theorem 3.6). This example shows how condition (3) of Theorem 3.6 can properly discriminate between the two cases illustrated in Figures 2(b) and 2(c).

Consider the polyhedra $\mathcal{Q}_3$ and $\mathcal{Q}_4$ in Figure 2(b), where $\mathcal{Q}_3$ is the rectangle $ABCD$ with the open bound (B,C) defined by the strict constraint $x_1 < 3$, whereas $\mathcal{Q}_4$ is the rectangle $BEFC$ with the open bound (B,C) defined by the strict constraint $x_1 > 3$. Thus $B = (3,1)^{\mathrm{T}}$ and $C = (3,5)^{\mathrm{T}}$ are closure points for both $\mathcal{Q}_3$ and $\mathcal{Q}_4$. It can be seen that $\mathcal{Q}_3 \uplus \mathcal{Q}_4$, the polyhedron in Figure 2(e), contains the whole line segment $[B,C]$ so that $\mathcal{Q}_3 \uplus \mathcal{Q}_4 \neq \mathcal{Q}_3 \cup \mathcal{Q}_4$. In the statement of Theorem 3.6, let $\mathcal{P}_1 = \mathcal{Q}_3$, $\mathcal{P}_2 = \mathcal{Q}_4$, $i = 1$, $j = 2$, $\beta_1 = (x_1 < 3) \in \mathcal{C}_1$ and $g_1 = B$ be a closure point in $\mathcal{G}_1$. Then β_1 is violated by $\mathcal{P}_2$ and saturated by the closure point g_1. Although condition (1) does not hold because g_1 is subsumed by $\mathcal{P}_2$, condition (3) does hold since β_1 is strict and $g_1 \in (\mathcal{P}_1 \uplus \mathcal{P}_2) \setminus \mathcal{P}_2$.

Consider the polyhedra $\mathcal{Q}_5$ and $\mathcal{Q}_6$ in Figure 2(c), where $\mathcal{Q}_5$ is the quadrilateral $ABCD$ and $\mathcal{Q}_6$ is the quadrilateral $EFGC$. Then the convex polyhedral hull $\mathcal{Q}_5 \uplus \mathcal{Q}_6$ shown in Figure 2(f) is equal to their union $\mathcal{Q}_5 \cup \mathcal{Q}_6$. In the statement of Theorem 3.6, let $\mathcal{P}_1 = \mathcal{Q}_5$, $\mathcal{P}_2 = \mathcal{Q}_6$, $i = 1$, $j = 2$, $\beta_1 \in \mathcal{C}_1$ be the strict constraint defining the dashed line boundary (B, C) and g_1 be the closure point C in both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then none of the conditions in Theorem 3.6 hold.

4. Exact Join Detection for Boxes and Other Cartesian Products

A rational interval constraint for a dimension $i \in \{1, \dots, n\}$ has the form $x_i \bowtie b$, where $\bowtie \in \{<, \leq, =, \geq, >\}$ and $b \in \mathbb{Q}$. A finite system of rational interval constraints defines an NNC polyhedron in $\mathbb{P}_n$ that we call a *rational box*; the set of all rational boxes in the n-dimensional vector space is denoted $\mathbb{B}_n$ and is a meet-sublattice of $\mathbb{P}_n$. The domain $\mathbb{B}_n$ so defined can be seen as the Cartesian product of n possibly infinite intervals with rational, possibly open boundaries. If we denote by $\mathbb{I}$ the set of such intervals and by '$\oplus$' the binary join operator over the bounded join-semilattice $(\mathbb{I}, \subseteq)$, we have, for each $B_1, B_2 \in \mathbb{B}$,

$$B_1 \uplus B_2 = \bigl(\pi_1(B_1) \oplus \pi_1(B_2)\bigr) \times \cdots \times \bigl(\pi_n(B_1) \oplus \pi_n(B_2)\bigr).$$

The following theorem defines a necessary and sufficient condition that is only based on '$\oplus$' and on the subset ordering over $\mathbb{I}$. Notice, in particular, that convexity does not play any role, neither in the statement, nor in the proof.

Theorem 4.1. *Let $B_1, B_2 \in \mathbb{B}_n$. Then $B_1 \uplus B_2 \neq B_1 \cup B_2$ if and only if*

1. $\exists i \in \{1, \dots, n\} \mathrel{.} \pi_i(B_1) \oplus \pi_i(B_2) \neq \pi_i(B_1) \cup \pi_i(B_2)$*; or*
2. $\exists i, j \in \{1, \dots, n\} \mathrel{.} i \neq j \wedge \pi_i(B_1) \nsubseteq \pi_i(B_2) \wedge \pi_j(B_2) \nsubseteq \pi_j(B_1)$.

Proof. Suppose that $B_1 = \varnothing$ so that, for each $i \in \{1, \dots, n\}$, $\pi_i(B_1) = \varnothing$. Then, neither condition (1) nor condition (2) can hold, so that the lemma holds. By a symmetric reasoning, the lemma holds if $B_2 = \varnothing$. Hence, in the following we assume that both B_1 and B_2 are non-empty boxes.

Suppose first that $B_1 \uplus B_2 \neq B_1 \cup B_2$; then there exists a point $\mathbf{p} \in B_1 \uplus B_2$ such that $\mathbf{p} \notin B_1$ and $\mathbf{p} \notin B_2$. Hence, for some $i, j \in \{1, \dots, n\}$, we have that $p_i \notin \pi_i(B_1)$ and $p_j \notin \pi_j(B_2)$. Note that as $\mathbf{p} \in B_1 \uplus B_2$, we also have $p_i \in \pi_i(B_1) \oplus \pi_i(B_2)$ and $p_j \in \pi_j(B_1) \oplus \pi_j(B_2)$. Suppose that condition (1) does not hold. Then $p_i \in \pi_i(B_2)$ and $p_j \in \pi_j(B_1)$; hence we must have $i \neq j$ and $p_i \in \pi_i(B_1) \setminus \pi_i(B_2)$ and $p_j \in \pi_j(B_2) \setminus \pi_j(B_1)$; implying that $\pi_i(B_1) \nsubseteq \pi_i(B_2)$ and $\pi_j(B_2) \nsubseteq \pi_j(B_1)$, so that condition (2) holds.

Assuming that condition (1) or (2) holds, we now prove $B_1 \uplus B_2 \neq B_1 \cup B_2$. First, suppose that condition (1) holds. Then there exists $v \in \pi_i(B_1 \uplus B_2)$ such that $v \notin \pi_i(B_1)$ and $v \notin \pi_i(B_2)$. By definition of π_i, there exist a point $\mathbf{p} \in B_1 \uplus B_2$ such that $\pi_i(\mathbf{p}) = v$, so that $\mathbf{p} \notin B_1$ and $\mathbf{p} \notin B_2$; therefore $B_1 \uplus B_2 \neq B_1 \cup B_2$. Secondly, suppose that condition (2) holds. Then there exist values $v_i \in \pi_i(B_1) \setminus \pi_i(B_2)$ and $v_j \in \pi_j(B_2) \setminus \pi_j(B_1)$; hence, there exist points $\mathbf{p}_i \in B_1$ and $\mathbf{p}_j \in B_2$ such that $\pi_i(\mathbf{p}_i) = v_i$ and $\pi_j(\mathbf{p}_j) = v_j$. Let $\mathbf{p}$ be such that $\pi_k(\mathbf{p}) = \pi_k(\mathbf{p}_i)$, for all $k \in \{1, \dots, n\} \setminus \{j\}$, and $\pi_j(\mathbf{p}) = v_j$; then $\mathbf{p} \notin B_1 \cup B_2$. By definition of the '$\uplus$' operator, $\mathbf{p} \in B_1 \uplus B_2$, so that $B_1 \uplus B_2 \neq B_1 \cup B_2$. $\square$

Example 4.2. Consider the topologically closed boxes

$$B_1 = \operatorname{con}\bigl(\{0 \le x_1 \le 1, 0 \le x_2 \le 2\}\bigr),$$
$$B_2 = \operatorname{con}\bigl(\{3 \le x_1 \le 4, 0 \le x_2 \le 2\}\bigr),$$
$$B_3 = \operatorname{con}\bigl(\{0 \le x_1 \le 4, 1 \le x_2 \le 2\}\bigr).$$

Then we obtain

$$B_1 \uplus B_2 = B_1 \uplus B_3 = \operatorname{con}\bigl(\{0 \le x_1 \le 4, 0 \le x_2 \le 2\}\bigr).$$

Letting $\mathbf{p} = (2,0)^{\mathrm{T}}$, we have $\mathbf{p} \in B_1 \uplus B_2$ although $\mathbf{p} \notin B_1 \cup B_2 \cup B_3$; hence $B_1 \uplus B_2 \neq B_1 \cup B_2$ and $B_1 \uplus B_3 \neq B_1 \cup B_3$, i.e., both join computations are inexact. Observe that

$$\pi_1(B_1) \oplus \pi_1(B_2) \neq \pi_1(B_1) \cup \pi_1(B_2),$$

so that, for boxes B_1 and B_2, condition (1) holds; on the other hand we have

$$\pi_1(B_3) \nsubseteq \pi_1(B_1) \quad \text{and} \quad \pi_2(B_1) \nsubseteq \pi_2(B_3),$$

so that, for boxes B_1 and B_3, condition (2) holds.

This result has been introduced for rational boxes for simplicity only. Indeed, it trivially generalizes to any Cartesian product of 1-dimensional numerical abstractions, including: the well-known abstract domain of multi-dimensional, integer-valued intervals [20]; 1-dimensional congruence equations like $x = 0 \pmod 2$; *modulo intervals* [33, 34]; and *circular linear progressions* [35]. For full generality, for each $i \in \{1, \ldots, n\}$, let $(\mathbb{A}(i), \subseteq)$, with $\varnothing \in \mathbb{A}(i) \subseteq \wp(\mathbb{R})$, be a bounded join-semilattice where the binary join operator is denoted by '$\oplus_i$'. $(\mathbb{A}(i), \subseteq)$ is thus an abstract domain suitable for approximating $\wp(\mathbb{R})$. Then, the trivial combination of the n domains $\mathbb{A}(i)$ by means of Cartesian product, $\mathbb{A}_n := \mathbb{A}(1) \times \cdots \times \mathbb{A}(n)$, is an abstract domain suitable for approximating $\wp(\mathbb{R}^n)$.[5] Theorem 4.1 immediately generalizes to any domain $\mathbb{A}_n$ so obtained.

An algorithm for the exact join detection on $\mathbb{A}_n$ based on Theorem 4.1 will compute, in the worst case, a linear number of 1-dimensional joins (applying the '$\oplus_i$' operators) and a quadratic number of 1-dimensional inclusion tests. Since these 1-dimensional operations take constant time, the worst-case complexity bound for n-dimensional boxes is $\mathrm{O}(n^2)$.

5. Exact Join Detection for Bounded Difference Shapes

A (rational) bounded difference is a non-strict inequality constraint having one of the forms $\pm x_i \le b$ or $x_i - x_j \le b$, where $i, j \in \{1, \ldots, n\}$, $i \neq j$ and $b \in \mathbb{Q}$.

[5] This construction is called a *direct product* in the field of abstract interpretation. The resulting domain is said to be *attribute-independent*, in the sense that relational information is not captured. In other words, the constraints on space dimension i are unrelated to those on space dimension j whenever $i \neq j$.

A finite system of bounded differences defines a *bounded difference shape* (BD shape); the set of all BD shapes in the n-dimensional vector space is denoted $\mathbb{BD}_n$ and it is a meet-sublattice of $\mathbb{CP}_n$. In this section we specialize the result on topologically closed polyhedra to the case of BD shapes, which can be efficiently represented and manipulated as weighted graphs. To start with, we introduce some notation and terminology (see also [2, 9, 30, 31]).

Let $\mathbb{Q}_\infty := \mathbb{Q} \cup \{+\infty\}$ be totally ordered by the extension of '$<$' such that $d < +\infty$ for each $d \in \mathbb{Q}$. Let $\mathcal{N}$ be a finite set of *nodes*. A *weighted directed graph* (graph, for short) G in $\mathcal{N}$ is a pair $(\mathcal{N}, w)$, where $w \colon \mathcal{N} \times \mathcal{N} \to \mathbb{Q}_\infty$ is the weight function for G. A pair $(n_i, n_j) \in \mathcal{N} \times \mathcal{N}$ is an *arc* of G if $w(n_i, n_j) < +\infty$; the arc is *proper* if $n_i \neq n_j$. A *path* $\theta = n_0 \cdots n_p$ in a graph $G = (\mathcal{N}, w)$ is a non-empty and finite sequence of nodes such that, for all $i \in \{1, \dots, p\}$, (n_{i-1}, n_i) is an arc of G; each arc (n_{i-1}, n_i) is said to be *in* the path θ. If $\theta_1 = n_0 \cdots n_h$ and $\theta_2 = n_h \cdots n_p$ are paths in G, where $0 \leq h \leq p$, then the path concatenation $\theta = n_0 \cdots n_h \cdots n_p$ of θ_1 and θ_2 is denoted by $\theta_1 :: \theta_2$; if $\theta_1 = n_0 n_1$ (so that $h = 1$), then $\theta_1 :: \theta_2$ will also be denoted by $n_0 \cdot \theta_2$. Note that path concatenation is not the same as sequence concatenation. The path θ is *simple* if each node occurs at most once in θ; it is *proper* if all the arcs in it are proper; it is a *proper cycle* if it is a proper path and $n_0 = n_p$ (so that $p \geq 2$). The path θ has *weight* $w(\theta) := \sum_{i=1}^{p} w(n_{i-1}, n_i)$. A graph is *consistent* if it has no negative weight cycles. The set $\mathbb{G}$ of consistent graphs in $\mathcal{N}$ is partially ordered by the relation '$\trianglelefteq$' defined, for all $G_1 = (\mathcal{N}, w_1)$ and $G_2 = (\mathcal{N}, w_2)$, by

$$G_1 \trianglelefteq G_2 \quad \iff \quad \forall i, j \in \mathcal{N} : w_1(i, j) \leq w_2(i, j).$$

When augmented with a bottom element $\bot$ representing inconsistency, this partially ordered set becomes a (non-complete) lattice $\mathbb{G}_\bot = \langle \mathbb{G} \cup \{\bot\}, \trianglelefteq, \sqcap, \sqcup \rangle$, where '$\sqcap$' and '$\sqcup$' denote the (finitary) greatest lower bound and least upper bound operators, respectively.

Definition 5.1. (Graph closure/reduction.) A consistent graph $G = (\mathcal{N}, w)$ is *(shortest-path) closed* if the following properties hold:

$$\forall i \in \mathcal{N} : w(i, i) = 0; \tag{3}$$
$$\forall i, j, k \in \mathcal{N} : w(i, j) \leq w(i, k) + w(k, j). \tag{4}$$

The *closure* of a consistent graph G in $\mathcal{N}$ is

$$\operatorname{closure}(G) := \bigsqcup \bigl\{ G^{\mathrm{c}} \in \mathbb{G} \bigm| G^{\mathrm{c}} \trianglelefteq G \text{ and } G^{\mathrm{c}} \text{ is closed} \bigr\}.$$

A consistent graph R in $\mathcal{N}$ is *(shortest-path) reduced* if, for each graph $G \neq R$ such that $R \trianglelefteq G$, $\operatorname{closure}(R) \neq \operatorname{closure}(G)$. A *reduction* for the consistent graph G is any reduced graph R such that $\operatorname{closure}(R) = \operatorname{closure}(G)$.

Note that a reduction R for a closed graph G is a *subgraph* of G, meaning that all the arcs in R are also arcs in G and have the same finite weight.

Any non-empty element $\mathrm{bd} \in \mathbb{BD}_n$ can be represented by a consistent graph $G = (\mathcal{N}, w)$, where $\mathcal{N} = \{0, \dots, n\}$ is the set of graph nodes; each node $i > 0$

corresponds to the space dimension x_i of the vector space, while 0 (the *special node*) represents a further space dimension whose value is fixed to zero. Graph closure provides a normal form for non-empty BD shapes. Informally, a closed (resp., reduced) graph encodes a system of bounded difference constraints which is closed by entailment (resp., contains no redundant constraint).

If the non-empty BD shapes $\mathrm{bd}_1, \mathrm{bd}_2 \in \mathbb{BD}_n$ are represented by closed graphs $G_1 = (\mathcal{N}, w_1)$ and $G_2 = (\mathcal{N}, w_2)$, respectively, then the BD shape join $\mathrm{bd}_1 \uplus \mathrm{bd}_2$ is represented by the graph least upper bound $G_1 \sqcup G_2 = (\mathcal{N}, w)$, where $w(i,j) := \max\bigl(w_1(i,j), w_2(i,j)\bigr)$ for each $i, j \in \mathcal{N}$; $G_1 \sqcup G_2$ is also closed. Observe too that the set intersection $\mathrm{bd}_1 \cap \mathrm{bd}_2$ is represented by the graph greatest lower bound $G_1 \sqcap G_2$.

Theorem 5.2. *For each $h \in \{1, 2\}$, let $\mathrm{bd}_h \in \mathbb{BD}_n$ be a non-empty BD shape represented by the closed graph $G_h = (\mathcal{N}, w_h)$ and let R_h be a subgraph of G_h such that* $\mathrm{closure}(R_h) = G_h$*. Let also $G_1 \sqcup G_2 = (\mathcal{N}, w)$. Then* $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$ *if and only if there exist arcs (i, j) of R_1 and (k, ℓ) of R_2 such that*

(1) *$w_1(i, j) < w_2(i, j)$ and $w_2(k, \ell) < w_1(k, \ell)$; and*

(2) *$w_1(i, j) + w_2(k, \ell) < w(i, \ell) + w(k, j)$.*

Proof. Suppose that $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$, so that there exists $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$ such that $\mathbf{p} \notin \mathrm{bd}_1$ and $\mathbf{p} \notin \mathrm{bd}_2$. Hence, there exist $i, j, k, \ell \in \mathcal{N}$ such that (i, j) is an arc of R_1 satisfying[6] $\pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) > w_1(i, j)$ and (k, ℓ) is an arc of R_2 satisfying $\pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) > w_2(k, \ell)$. However, as $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$, $\pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) \leq w(i, j)$ and $\pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) \leq w(k, \ell)$ so that, by definition of $G_1 \sqcup G_2$, we have $w_1(i, j) < w_2(i, j)$ and $w_2(k, \ell) < w_1(k, \ell)$; hence condition (1) holds. Since $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$,

$$\begin{aligned} w(i, \ell) + w(k, j) &\geq \pi_i(\mathbf{p}) - \pi_\ell(\mathbf{p}) + \pi_k(\mathbf{p}) - \pi_j(\mathbf{p}) \\ &= \pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) + \pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) \\ &> w_1(i, j) + w_2(k, \ell). \end{aligned}$$

Therefore, condition (2) also holds.

We now suppose that there exist arcs (i, j) of R_1 and (k, ℓ) of R_2 such that conditions (1) and (2) hold. As G_1 and G_2 are closed, $w_1(i, i) = w_2(i, i) = 0$ and $w_1(k, k) = w_2(k, k) = 0$ so that condition (1) implies $i \neq j$ and $k \neq \ell$. As $G_1 \sqcup G_2$ is closed, $w(i, i) = w(k, k) = 0$ so that, if $i = \ell$ and $j = k$ both hold, condition (2) implies $w_1(i, j) + w_2(j, i) < 0$; hence, by condition (1) the graph greatest lower bound $G_1 \sqcap G_2$ contains the negative weight proper cycle $i \cdot j \cdot i$ and thus is inconsistent; hence $\mathrm{bd}_1 \cap \mathrm{bd}_2 = \varnothing$; and hence $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$. Therefore, in the following we assume that $i \neq \ell$ or $j \neq k$ hold. If the right hand side of the inequalities in conditions (1) and (2) are all unbounded, let $\epsilon := 1$;

[6] We extend notation by letting $\pi_0(\mathbf{v}) := 0$, for each vector $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{T}}$.

otherwise let

$$\epsilon := \min \left\{ \begin{array}{l} w(i,j) - w_1(i,j), \\ w(k,\ell) - w_2(k,\ell), \\ \frac{1}{2}\big(w(i,\ell) + w(k,j) - w_1(i,j) - w_2(k,\ell)\big) \end{array} \right\}.$$

Then, by conditions (1) and (2), $\epsilon > 0$. Consider the graph $G' = (\mathcal{N}, w')$ where, for each $r, s \in \mathcal{N}$,

$$w'(r,s) := \begin{cases} -w_1(i,j) - \epsilon, & \text{if } (r,s) = (j,i); \\ -w_2(k,\ell) - \epsilon, & \text{if } (r,s) = (\ell,k); \\ w(r,s), & \text{otherwise.} \end{cases}$$

We show that G' is a consistent graph; to this end, since $G := G_1 \sqcup G_2$ is known to be consistent, it is sufficient to consider the proper cycles of G' that contain arcs (j,i) or (ℓ,k). Let $\theta_{ij} = i \cdots j$ and $\theta_{k\ell} = k \cdots \ell$ be arbitrary simple paths from i to j and from k to ℓ, respectively. Then G' is consistent if and only if $w'(\theta_{ij} \cdot i) \geq 0$ and $w'(\theta_{k\ell} \cdot k) \geq 0$. We only prove $w'(\theta_{ij} \cdot i) \geq 0$ since the proof that $w'(\theta_{k\ell} \cdot k) \geq 0$ follows by a symmetrical argument. As θ_{ij} is simple, it does not contain the arc (j,i). Suppose first that θ_{ij} does not contain the arc (ℓ,k). Then

$$\begin{aligned} w'(\theta_{ij} \cdot i) &= w'(\theta_{ij}) + w'(j,i) \\ &= w(\theta_{ij}) - w_1(i,j) - \epsilon && [\text{def. of } w'] \\ &\geq w(i,j) - w_1(i,j) - \epsilon && [G \text{ closed}] \\ &\geq 0 && [\text{def. of } \epsilon]. \end{aligned}$$

Suppose now that $\theta_{ij} = \theta_{i\ell} :: (\ell,k) :: \theta_{kj}$, where $\theta_{i\ell} = i \cdots \ell$ and $\theta_{kj} = k \cdots j$ do not contain the arcs (j,i) and (k,ℓ). Then

$$\begin{aligned} w'(\theta_{ij} \cdot i) &= w'(\theta_{i\ell}) + w'(\ell,k) + w'(\theta_{kj}) + w'(j,i) \\ &= w(\theta_{i\ell}) - w_2(k,\ell) - \epsilon + w(\theta_{kj}) - w_1(i,j) - \epsilon && [\text{def. of } w'] \\ &\geq w(i,\ell) - w_2(k,\ell) - \epsilon + w(k,j) - w_1(i,j) - \epsilon && [G \text{ closed}] \\ &= \big(w(i,\ell) + w(k,j) - w_1(i,j) - w_2(k,\ell)\big) - 2\epsilon \\ &\geq 0 && [\text{def. of } \epsilon]. \end{aligned}$$

Therefore G' is consistent. Moreover, $G' \trianglelefteq G$ since

$$\begin{aligned} w'(j,i) &= -w_1(i,j) - \epsilon && [\text{def. of } w'] \\ &\leq -w_1(i,j) && [\epsilon \geq 0] \\ &\leq w_1(j,i) && [G_1 \text{ consistent}] \\ &\leq w(j,i) && [\text{def. } G]; \end{aligned}$$

similarly, $w'(\ell,k) \leq w(\ell,k)$; hence, for all $r, s \in \mathcal{N}$, $w'(r,s) \leq w(r,s)$.

Let $\mathrm{bd}' \in \mathbb{BD}_n$ be represented by G', so that $\varnothing \neq \mathrm{bd}' \subseteq \mathrm{bd}_1 \uplus \mathrm{bd}_2$. Since $w'(j,i) + w_1(i,j) < 0$, we obtain $\mathrm{bd}' \cap \mathrm{bd}_1 = \varnothing$; since $w'(\ell,k) + w_2(k,\ell) < 0$, we obtain $\mathrm{bd}' \cap \mathrm{bd}_2 = \varnothing$. Hence, $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$. □

An algorithm for the exact join detection on $\mathbb{BD}_n$ based on Theorem 5.2 will have a worst-case complexity bound in $\mathrm{O}(n^4)$. Noting that the computation of graph closure and reduction are both in $\mathrm{O}(n^3)$ [2, 9, 28, 31], a more detailed complexity bound is $\mathrm{O}(n^3 + r_1 r_2)$, where r_h is the number of arcs in the subgraph R_h; hence, a good choice is to take each R_h to be a graph reduction for G_h, as it will have a minimal number of arcs.

5.1. Integer BD Shapes

We now consider the case of integer BD shapes, i.e., subsets of $\mathbb{Z}^n$ that are delimited by BD constraints where the bounds are all integral. As for the rational case, these numerical abstractions can be encoded using weighted graphs, but restricting the codomain of the weight function to $\mathbb{Z}_\infty := \mathbb{Z} \cup \{+\infty\}$. Since the set of integer graphs is a sub-lattice of the set of rational graphs, the conditions in Theorem 5.2 can be easily strengthened so as to obtain the corresponding result for integer BD shapes. The algorithm for the integer case shares the same complexity bound of the rational case.

Theorem 5.3. *For each $h \in \{1,2\}$, let $\mathrm{bd}_h \in \mathbb{BD}_n$ be a non-empty integer BD shape represented by the closed integer graph $G_h = (\mathcal{N}, w_h)$ and let R_h be a subgraph of G_h such that* $\mathrm{closure}(R_h) = G_h$. *Let also $G_1 \sqcup G_2 = (\mathcal{N}, w)$. Then* $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$ *if and only if there exist arcs (i,j) of R_1 and (k,ℓ) of R_2 such that*

(1) $w_1(i,j) < w_2(i,j)$ *and* $w_2(k,\ell) < w_1(k,\ell)$*; and*

(2) $w_1(i,j) + w_2(k,\ell) + 2 \leq w(i,\ell) + w(k,j)$.

Proof. Suppose first that $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$, so that there exists $\mathbf{p} \in \mathbb{Z}^n$ such that $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$ but $\mathbf{p} \notin \mathrm{bd}_1$ and $\mathbf{p} \notin \mathrm{bd}_2$. Hence, there exist $i, j, k, \ell \in \mathcal{N}$ such that (i,j) is an arc of R_1 satisfying $\pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) > w_1(i,j)$ and (k,ℓ) is an arc of R_2 satisfying $\pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) > w_2(k,\ell)$. However, as $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$, $\pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) \leq w(i,j)$ and $\pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) \leq w(k,\ell)$ so that, by definition of $G_1 \sqcup G_2$, we have $w_1(i,j) < w_2(i,j)$ and $w_2(k,\ell) < w_1(k,\ell)$; hence condition (1) holds. Note also that $w_1(i,j)$ and $w_2(k,\ell)$ are both finite and hence in $\mathbb{Z}$ so that $w_1(i,j) \leq w_2(i,j) + 1$ and $w_2(k,\ell) \leq w_1(k,\ell) + 1$. Since $\mathbf{p} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$,

$$\begin{aligned} w(i,\ell) + w(k,j) &\geq \pi_i(\mathbf{p}) - \pi_\ell(\mathbf{p}) + \pi_k(\mathbf{p}) - \pi_j(\mathbf{p}) \\ &= \pi_i(\mathbf{p}) - \pi_j(\mathbf{p}) + \pi_k(\mathbf{p}) - \pi_\ell(\mathbf{p}) \\ &\geq w_1(i,j) + w_2(k,\ell) + 2. \end{aligned}$$

Therefore, condition (2) also holds.

We now suppose that there exist arcs (i,j) of R_1 and (k,ℓ) of R_2 such that conditions (1) and (2) hold. Let $G' = (\mathcal{N}, w')$ be a graph defined as in the proof of Theorem 5.2, where however we just define $\epsilon := 1$, so that G' is an integer graph. By using the same reasoning as in the proof of Theorem 5.2, it can be seen that G' is consistent and $G' \trianglelefteq G_1 \sqcup G_2$. Let $\mathrm{bd}' \in \mathbb{BD}_n$ be represented by G', so that $\varnothing \neq \mathrm{bd}' \subseteq \mathrm{bd}_1 \uplus \mathrm{bd}_2$. Since $w'(j,i) + w_1(i,j) < 0$, we obtain $\mathrm{bd}' \cap \mathrm{bd}_1 = \varnothing$; since $w'(\ell,k) + w_2(k,\ell) < 0$, we obtain $\mathrm{bd}' \cap \mathrm{bd}_2 = \varnothing$. Hence, $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2$. □

Example 5.4. Consider the 2-dimensional BD shapes

$$\begin{aligned}
\mathrm{bd}_1 &= \mathrm{con}\big(\{0 \leq x_1 \leq 3, 0 \leq x_2 \leq 2, x_1 - x_2 \leq 2\}\big),\\
\mathrm{bd}_2 &= \mathrm{con}\big(\{3 \leq x_1 \leq 6, 0 \leq x_2 \leq 2\}\big).
\end{aligned}$$

Then, in the case of rational BD shapes, the join $\mathrm{bd}_1 \uplus \mathrm{bd}_2$ is not exact, since $(2.5, 0)^{\mathrm{T}} \in \mathrm{bd}_1 \uplus \mathrm{bd}_2$ although $(2.5, 0)^{\mathrm{T}} \notin \mathrm{bd}_1 \cup \mathrm{bd}_2$. Taking $i = 1$, $j = 2$, $k = 0$ and $\ell = 1$, it can be seen that both conditions in Theorem 5.2 are satisfied; in particular, for the second condition we have

$$w_1(1,2) + w_2(0,1) = 2 - 3 \leq 0 + 0 = w(1,1) + w(0,2).$$

By contrast, in the case of integer BD shapes, the join is exact; in particular, with the above choice for indices i, j, k, ℓ, the second condition of Theorem 5.3 does not hold:

$$w_1(1,2) + w_2(0,1) + 2 = 2 - 3 + 2 > 0 + 0 = w(1,1) + w(0,2).$$

5.2. Generalizing this result to k BD shapes

We conjecture that the above results for the exact join detection of two (rational or integer) BD shapes can be generalized to any number of component BD shapes. That is, given k BD shapes $\mathrm{bd}_1, \ldots, \mathrm{bd}_k \in \mathbb{BD}_n$, it is possible to provide a suitable set of conditions that determine whether or not $\mathrm{bd}_1 \uplus \cdots \uplus \mathrm{bd}_k = \mathrm{bd}_1 \cup \cdots \cup \mathrm{bd}_k$. Here we just present the conjecture, for the rational case, when $k = 3$.

Conjecture 5.5. *For each $h \in \{1,2,3\}$, let $\mathrm{bd}_h \in \mathbb{BD}_n$ be a non-empty BD shape represented by the closed graph $G_h = (\mathcal{N}, w_h)$ and let R_h be a subgraph of G_h such that* $\mathrm{closure}(R_h) = G_h$. *Let also $G_1 \sqcup G_2 \sqcup G_3 = (\mathcal{N}, w)$. Then $\mathrm{bd}_1 \uplus \mathrm{bd}_2 \uplus \mathrm{bd}_3 \neq \mathrm{bd}_1 \cup \mathrm{bd}_2 \cup \mathrm{bd}_3$ if and only if there exist arcs (i_1, j_1) of R_1, (i_2, j_2) of R_2 and (i_3, j_3) of R_3, respectively, such that*

(1) *for each $h \in \{1,2,3\}$, $w_h(i_h, j_h) < w(i_h, j_h)$;*

(2a) *$w_1(i_1, j_1) + w_2(i_2, j_2) < w(i_1, j_2) + w(i_2, j_1)$;*

(2b) *$w_2(i_2, j_2) + w_3(i_3, j_3) < w(i_2, j_3) + w(i_3, j_2)$;*

(2c) *$w_3(i_3, j_3) + w_1(i_1, j_1) < w(i_3, j_1) + w(i_1, j_3)$;*

(3a) *$w_1(i_1, j_1) + w_2(i_2, j_2) + w_3(i_3, j_3) < w(i_1, j_2) + w(i_2, j_3) + w(i_3, j_1)$;*

(3b) $w_1(i_1, j_1) + w_2(i_2, j_2) + w_3(i_3, j_3) < w(i_1, j_3) + w(i_2, j_1) + w(i_3, j_2)$.

Even though the generalization is straightforward from a mathematical point of view, for larger values of k this will result in having to check a rather involved combinatorial combination of all the conditions.

6. Exact Join Detection for Octagonal Shapes

Octagonal constraints generalize BD constraints by also allowing for non-strict inequalities having the form $x_i + x_j \leq b$ or $-x_i - x_j \leq b$. Octagonal constraints can be encoded using BD constraints by splitting each variable x_i into two forms: a positive form x_i^+, interpreted as $+x_i$; and a negative form x_i^-, interpreted as $-x_i$. For instance, an octagonal constraint such as $x_i + x_j \leq b$ can be translated into the potential constraint $x_i^+ - x_j^- \leq b$; alternatively, the same octagonal constraint can be translated into $x_j^+ - x_i^- \leq b$. Unary (octagonal) constraints such as $x_i \leq b$ and $-x_i \leq b$ are encoded as $x_i^+ - x_i^- \leq 2b$ and $x_i^- - x_i^+ \leq -2b$, respectively.

From now on, we assume that the set of nodes is $\mathcal{N} := \{0, \dots, 2n-1\}$. These will denote the positive and negative forms of the vector space dimensions x_1, …, x_n: for all $i \in \mathcal{N}$, if $i = 2k$, then i represents the positive form x_{k+1}^+ and, if $i = 2k + 1$, then i represents the negative form x_{k+1}^- of the dimension x_{k+1}. To simplify the presentation, we let $\bar{\imath}$ denote $i + 1$, if i is even, and $i - 1$, if i is odd, so that, for all $i \in \mathcal{N}$, we also have $\bar{\imath} \in \mathcal{N}$ and $\bar{\bar{\imath}} = i$.

It follows from the above translations that any finite system of octagonal constraints, translated into a set of potential constraints in $\mathcal{N}$ as above, can be encoded by a graph G in $\mathcal{N}$. In particular, any finite *satisfiable* system of octagonal constraints can be encoded by a *consistent* graph in $\mathcal{N}$. However, the converse does not hold since in any valuation ρ of an encoding of a set of octagonal constraints we must also have $\rho(i) = -\rho(\bar{\imath})$, so that the arcs (i, j) and $(\bar{\jmath}, \bar{\imath})$ should have the same weight. Therefore, to encode rational octagonal constraints, we restrict attention to consistent graphs over $\mathcal{N}$ where the arcs in all such pairs are *coherent*.

Definition 6.1. (Octagonal graph.) A (rational) *octagonal graph* is any consistent graph $G = (\mathcal{N}, w)$ that satisfies the coherence assumption:

$$\forall i, j \in \mathcal{N} : w(i, j) = w(\bar{\jmath}, \bar{\imath}). \tag{5}$$

The set $\mathbb{O}$ of all octagonal graphs (with the usual addition of the bottom element, representing an unsatisfiable system of constraints) is a sub-lattice of $\mathbb{G}_\perp$, sharing the same least upper bound and greatest lower bound operators. Note that, at the implementation level, coherence can be automatically and efficiently enforced by letting arc (i, j) and arc $(\bar{\jmath}, \bar{\imath})$ share the same representation.

The standard shortest-path closure algorithm is not enough to obtain a canonical form for octagonal graphs.

Definition 6.2. (Graph strong closure/reduction.) An octagonal graph $G = (\mathcal{N}, w)$ is *strongly closed* if it is closed and the following property holds:

$$\forall i, j \in \mathcal{N} : 2w(i, j) \leq w(i, \bar{\imath}) + w(\bar{\jmath}, j). \tag{6}$$

The *strong closure* of an octagonal graph G in $\mathcal{N}$ is

$$\text{S-closure}(G) := \bigsqcup \bigl\{ G' \in \mathbb{O} \bigm| G' \trianglelefteq G \text{ and } G' \text{ is strongly closed} \bigr\}.$$

An octagonal graph R is *strongly reduced* if, for each octagonal graph $G \neq R$ such that $R \trianglelefteq G$, we have $\text{S-closure}(R) \neq \text{S-closure}(G)$. A *strong reduction* for the octagonal graph G is any strongly reduced octagonal graph R such that $\text{S-closure}(R) = \text{S-closure}(G)$.

Observe that, as was the case for shortest-path reduction, a strong reduction for a strongly closed graph G is a subgraph of G.

We denote by $\mathbb{OCT}_n$ the domain of octagonal shapes, whose non-empty elements can be represented by octagonal graphs: $\mathbb{BD}_n$ is a meet-sublattice of $\mathbb{OCT}_n$, which in turn is a meet-sublattice of $\mathbb{CP}_n$. A strongly closed (resp., strongly reduced) graph encodes a system of octagonal constraints which is closed by entailment (resp., contains no redundant constraint).

Theorem 6.3. *For each $h \in \{1, 2\}$, let $\mathrm{oct}_h \in \mathbb{OCT}_n$ be a non-empty octagonal shape represented by the strongly closed graph $G_h = (\mathcal{N}, w_h)$ and let R_h be a subgraph of G_h such that $\text{S-closure}(R_h) = G_h$. Let also $G_1 \sqcup G_2 = (\mathcal{N}, w)$. Then $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$ if and only if there exist arcs (i, j) of R_1 and (k, ℓ) of R_2 such that*

(1a) $w_1(i, j) < w_2(i, j)$;

(1b) $w_2(k, \ell) < w_1(k, \ell)$;

(2a) $w_1(i, j) + w_2(k, \ell) < w(i, \ell) + w(k, j)$;

(2b) $w_1(i, j) + w_2(k, \ell) < w(i, \overline{k}) + w(\bar{\jmath}, \ell)$;

(3a) $2w_1(i, j) + w_2(k, \ell) < w(i, \ell) + w(i, \overline{k}) + w(\bar{\jmath}, j)$;

(3b) $2w_1(i, j) + w_2(k, \ell) < w(k, j) + w(\bar{\jmath}, \ell) + w(i, \bar{\imath})$;

(4a) $w_1(i, j) + 2w_2(k, \ell) < w(i, \ell) + w(\bar{\jmath}, \ell) + w(k, \overline{k})$;

(4b) $w_1(i, j) + 2w_2(k, \ell) < w(k, j) + w(i, \overline{k}) + w(\overline{\ell}, \ell)$.

Proof. For each $r \in \mathcal{N} = \{0, \ldots, 2n - 1\}$ and each $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{T}} \in \mathbb{R}^n$, we denote by $\tilde{\pi}_r(\mathbf{v})$ the projection of vector $\mathbf{v}$ on the space dimension corresponding to the octagonal graph node r, defined as:

$$\tilde{\pi}_r(\mathbf{v}) := \begin{cases} v_{s+1}, & \text{if } r = 2s; \\ -v_{s+1}, & \text{if } r = 2s + 1. \end{cases}$$

Suppose that $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$, so that there exists $\mathbf{p} \in \mathrm{oct}_1 \uplus \mathrm{oct}_2$ such that $\mathbf{p} \notin \mathrm{oct}_1$ and $\mathbf{p} \notin \mathrm{oct}_2$. Hence, there exist arcs (i, j) and (k, ℓ) of R_1 and R_2, respectively, satisfying

$$\begin{aligned} w(i,j) &\geq \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) > w_1(i,j), \\ w(k,\ell) &\geq \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) > w_2(k,\ell); \end{aligned}$$

hence conditions (1a) and (1b) hold;

$$\begin{aligned} w(i,\ell) + w(k,j) &\geq \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) \\ &= \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) \\ &> w_1(i,j) + w_2(k,\ell) \end{aligned}$$

so that condition (2a) holds and, by a symmetric argument, condition (2b) holds;

$$\begin{aligned} w(i,\ell) + w(i,\overline{k}) + w(\overline{\jmath},j) &\geq \bigl(\tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p})\bigr) + \bigl(\tilde{\pi}_i(\mathbf{p}) + \tilde{\pi}_k(\mathbf{p})\bigr) + \bigl(-2\,\tilde{\pi}_j(\mathbf{p})\bigr) \\ &= 2\bigl(\tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p})\bigr) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) \\ &> 2w_1(i,j) + w_2(k,\ell) \end{aligned}$$

so that condition (3a) holds; conditions (3b), (4a) and (4b) follow by symmetric arguments.

We now suppose that, for some $i, j, k, \ell \in \mathcal{N}$, all conditions (1a) – (4b) hold. Note that, by (1a) and (1b), $i \neq j$ and $k \neq \ell$. Suppose first that $(i, j) \in \bigl\{(\ell, k), (\overline{k}, \overline{\ell})\bigr\}$; then, conditions (1a) – (2b) imply $w_1(i, j) + w_2(j, i) < 0$, so that the graph greatest lower bound $G_1 \sqcap G_2$ is inconsistent, as it contains a negative weight proper cycle; hence, $\mathrm{oct}_1 \cap \mathrm{oct}_2 = \varnothing$, which implies $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$. Therefore, in the following we assume that $(i, j) \notin \bigl\{(\ell, k), (\overline{k}, \overline{\ell})\bigr\}$ holds.

If the right hand sides of the inequalities in conditions (1a) – (4b) are all unbounded, let $\epsilon := 1$; otherwise let

$$\epsilon := \min \left\{ \begin{array}{l} w(i,j) - w_1(i,j), \\ w(k,\ell) - w_2(k,\ell), \\ \dfrac{1}{2}\bigl(w(i,\ell) + w(k,j) - w_1(i,j) - w_2(k,\ell)\bigr), \\ \dfrac{1}{2}\bigl(w(i,\overline{k}) + w(\overline{\jmath},\ell) - w_1(i,j) - w_2(k,\ell)\bigr), \\ \dfrac{1}{3}\bigl(w(i,\ell) + w(i,\overline{k}) + w(\overline{\jmath},j) - 2w_1(i,j) - w_2(k,\ell)\bigr), \\ \dfrac{1}{3}\bigl(w(k,j) + w(\overline{\jmath},\ell) + w(i,\overline{\imath}) - 2w_1(i,j) - w_2(k,\ell)\bigr), \\ \dfrac{1}{3}\bigl(w(i,\ell) + w(\overline{\jmath},\ell) + w(k,\overline{k}) - w_1(i,j) - 2w_2(k,\ell)\bigr), \\ \dfrac{1}{3}\bigl(w(k,j) + w(i,\overline{k}) + w(\overline{\ell},\ell) - w_1(i,j) - 2w_2(k,\ell)\bigr) \end{array} \right\}.$$

Then, by conditions (1a) – (4b) $\epsilon > 0$. Consider the graph $G' = (\mathcal{N}, w')$ where, for each $r, s \in \mathcal{N}$,

$$w'(r,s) := \begin{cases} -w_1(i,j) - \epsilon, & \text{if } (r,s) \in \big\{(j,i), (\bar{\imath}, \bar{\jmath})\big\}; \\ -w_2(k,\ell) - \epsilon, & \text{if } (r,s) \in \big\{(\ell,k), (\overline{k}, \overline{\ell})\big\}; \\ w(r,s), & \text{otherwise.} \end{cases}$$

Let $G := G_1 \sqcup G_2$; as G is coherent, G' is coherent too. We now show that G' is a consistent graph; to this end, since G is known to be consistent, it is sufficient to consider the proper cycles of G' that contain arc (j,i) or arc (ℓ,k).[7] Let $\theta_{ij} = i \cdots j$ and $\theta_{k\ell} = k \cdots \ell$ be any simple paths from i to j and from k to ℓ, respectively. Then G' is consistent if and only if $w'(\theta_{ij} \cdot i) \geq 0$ and $w'(\theta_{k\ell} \cdot k) \geq 0$. We only prove $w'(\theta_{ij} \cdot i) \geq 0$ since the proof that $w'(\theta_{k\ell} \cdot k) \geq 0$ follows by a symmetrical argument. Since θ_{ij} is simple, it does not contain the arc (j,i). In the following we consider in detail five cases, again noting that all the other cases can be proved by symmetrical arguments:

1. θ_{ij} contains none of the arcs (ℓ,k), $(\overline{k}, \overline{\ell})$ and $(\bar{\imath}, \bar{\jmath})$;
2. $\theta_{ij} = \theta_{i\bar{\imath}} :: (\bar{\imath}, \bar{\jmath}) :: \theta_{\bar{\jmath}j}$;
3. $\theta_{ij} = \theta_{i\ell} :: (\ell,k) :: \theta_{kj}$;
4. $\theta_{ij} = \theta_{i\ell} :: (\ell,k) :: \theta_{k\overline{k}} :: (\overline{k}, \overline{\ell}) :: \theta_{\overline{\ell}j}$;
5. $\theta_{ij} = \theta_{i\ell} :: (\ell,k) :: \theta_{k\overline{k}} :: (\overline{k}, \overline{\ell}) :: \theta_{\overline{\ell}\bar{\imath}} :: (\bar{\imath}, \bar{\jmath}) :: \theta_{\bar{\jmath}j}$,

where the simple paths $\theta_{i\bar{\imath}}$, $\theta_{i\ell}$, θ_{kj}, $\theta_{k\overline{k}}$, $\theta_{\overline{\ell}j}$, $\theta_{\overline{\ell}\bar{\imath}}$ and $\theta_{\bar{\jmath}j}$ contain none of the arcs (ℓ,k), $(\overline{k}, \overline{\ell})$ and $(\bar{\imath}, \bar{\jmath})$.

- Case (1).

$$\begin{aligned} w'(\theta_{ij} \cdot i) &= w'(\theta_{ij}) + w'(j,i) \\ &= w(\theta_{ij}) - w_1(i,j) - \epsilon && [\text{def. of } w'] \\ &\geq w(i,j) - w_1(i,j) - \epsilon && [G \text{ closed}] \\ &\geq 0 && [\text{def. of } \epsilon]. \end{aligned}$$

[7] Any cycle containing arc $(\bar{\imath}, \bar{\jmath})$ (resp., $(\overline{k}, \overline{\ell})$) can be transformed to the corresponding coherent cycle containing arc (j,i) (resp., (ℓ,k)), having the same weight.

- Case (2).

$$\begin{aligned}
w'(\theta_{ij} \cdot i) &= w'(\theta_{i\overline{\imath}}) + w'(\overline{\imath}, \overline{\jmath}) + w'(\theta_{\overline{\jmath}j}) + w'(j,i) \\
&= w'(\theta_{i\overline{\imath}}) + w'(\theta_{\overline{\jmath}j}) + 2w'(j,i) && [G' \text{ coherent}] \\
&= w(\theta_{i\overline{\imath}}) + w(\theta_{\overline{\jmath}j}) - 2w_1(i,j) - 2\epsilon && [\text{def. of } w'] \\
&\geq w(i,\overline{\imath}) + w(\overline{\jmath},j) - 2w_1(i,j) - 2\epsilon && [G \text{ closed}] \\
&\geq 2w(i,j) - 2w_1(i,j) - 2\epsilon && [G \text{ strongly closed}] \\
&= 2\big(w(i,j) - w_1(i,j)\big) - 2\epsilon \\
&\geq 0 && [\text{def. of } \epsilon].
\end{aligned}$$

- Case (3).

$$\begin{aligned}
w'(\theta_{ij} \cdot i) &= w'(\theta_{i\ell}) + w'(\ell,k) + w'(\theta_{kj}) + w'(j,i) \\
&= w(\theta_{i\ell}) - w_2(k,\ell) - \epsilon + w(\theta_{kj}) - w_1(i,j) - \epsilon && [\text{def. of } w'] \\
&\geq w(i,\ell) - w_2(k,\ell) - \epsilon + w(k,j) - w_1(i,j) - \epsilon && [G \text{ closed}] \\
&= \big(w(i,\ell) + w(k,j) - w_1(i,j) - w_2(k,\ell)\big) - 2\epsilon \\
&\geq 0 && [\text{def. of } \epsilon].
\end{aligned}$$

- Case (4).

$$\begin{aligned}
&w'(\theta_{ij} \cdot i) \\
&= w'(\theta_{i\ell}) + w'(\ell,k) + w'(\theta_{k\overline{k}}) + w'(\overline{k},\overline{\ell}) + w'(\theta_{\overline{\ell}j}) + w'(j,i) \\
&= w'(\theta_{i\ell}) + 2w'(\ell,k) + w'(\theta_{k\overline{k}}) + w'(\theta_{\overline{\jmath}\ell}) + w'(j,i) && [G' \text{ coherent}] \\
&= w(\theta_{i\ell}) - 2w_2(k,\ell) - 2\epsilon + w(\theta_{k\overline{k}}) + w(\theta_{\overline{\jmath}\ell}) - w_1(i,j) - \epsilon && [\text{def. of } w'] \\
&\geq w(i,\ell) - 2w_2(k,\ell) - 2\epsilon + w(k,\overline{k}) + w(\overline{\jmath},\ell) - w_1(i,j) - \epsilon && [G \text{ closed}] \\
&= \big(w(i,\ell) + w(\overline{\jmath},\ell) + w(k,\overline{k}) - w_1(i,j) - 2w_2(k,\ell)\big) - 3\epsilon \\
&\geq 0 && [\text{def. of } \epsilon].
\end{aligned}$$

- Case (5).

$$
\begin{aligned}
w'(\theta_{ij} \cdot i) &= w'(\theta_{i\ell}) + w'(\ell, k) + w'(\theta_{k\overline{k}}) + w'(\overline{k}, \overline{\ell}) \\
&\quad + w'(\theta_{\overline{\ell}\overline{\imath}}) + w'(\overline{\imath}, \overline{\jmath}) + w'(\theta_{\overline{\jmath}j}) + w'(j, i) \\
&= 2w'(\theta_{i\ell}) + 2w'(j, i) \\
&\quad + w'(\theta_{k\overline{k}}) + w'(\theta_{\overline{\jmath}j}) + 2w'(\ell, k) && [G' \text{ coherent}] \\
&= 2w(\theta_{i\ell}) - 2w_1(i, j) - 2\epsilon \\
&\quad + w(\theta_{k\overline{k}}) + w(\theta_{\overline{\jmath}j}) - 2w_2(k, \ell) - 2\epsilon && [\text{def. of } w'] \\
&\geq 2w(i, \ell) - 2w_1(i, j) - 2\epsilon \\
&\quad + w(k, \overline{k}) + w(\overline{\jmath}, j) - 2w_2(k, \ell) - 2\epsilon && [G \text{ closed}] \\
&\geq 2w(i, \ell) - 2w_1(i, j) - 2\epsilon \\
&\quad + 2w(k, j) - 2w_2(k, \ell) - 2\epsilon && [G \text{ strongly closed}] \\
&= 2\big(w(i, \ell) + w(k, j) - w_1(i, j) - w_2(k, \ell)\big) - 4\epsilon \\
&\geq 0 && [\text{def. of } \epsilon].
\end{aligned}
$$

Therefore G' is consistent. Moreover, $G' \trianglelefteq G$ since

$$
\begin{aligned}
w'(j, i) &= -w_1(i, j) - \epsilon && [\text{def. of } w'] \\
&\leq -w_1(i, j) && [\epsilon \geq 0] \\
&\leq w_1(j, i) && [G_1 \text{ consistent}] \\
&\leq w(j, i) && [\text{def. } G];
\end{aligned}
$$

similarly, $w'(\ell, k) \leq w(\ell, k)$; hence, for all $r, s \in \mathcal{N}$, $w'(r, s) \leq w(r, s)$.

Let $\mathrm{oct}' \in \mathbb{OCT}_n$ be represented by G', so that $\varnothing \neq \mathrm{oct}' \subseteq \mathrm{oct}_1 \uplus \mathrm{oct}_2$. Since $w'(j, i) + w_1(i, j) < 0$, we obtain $\mathrm{oct}' \cap \mathrm{oct}_1 = \varnothing$; since $w'(\ell, k) + w_2(k, \ell) < 0$, we obtain $\mathrm{oct}' \cap \mathrm{oct}_2 = \varnothing$. Hence, $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$. ☐

Since the computation of the strong closure and strong reduction of an octagonal graph are both in $\mathrm{O}(n^3)$ [2, 9, 31], an algorithm for the exact join detection on $\mathbb{OCT}_n$ based on Theorem 6.3 has the same asymptotic worst-case complexity as the corresponding algorithm for $\mathbb{BD}_n$.

Example 6.4. Consider the 2-dimensional octagonal shapes

$$
\begin{aligned}
\mathrm{oct}_1 &= \mathrm{con}\big(\{x_1 + x_2 \leq 0\}\big), \\
\mathrm{oct}_2 &= \mathrm{con}\big(\{x_1 \leq 2\}\big).
\end{aligned}
$$

Then, the join $\mathrm{oct}_1 \uplus \mathrm{oct}_2 = \mathbb{R}^2$ is not exact. In fact, taking $i = 0$, $j = 3$, $k = 0$ and $\ell = 1$, we have $w_1(i, j) = 0$ and $w_2(k, \ell) = 4$, so that all the left hand sides in conditions (1a) – (4b) are finite; since all the corresponding right hand sides are infinite, all the conditions hold.

6.1. Integer Octagonal Shapes

We now consider the case of integer octagonal constraints, i.e., octagonal constraints where the bounds are all integral and the variables are only allowed to take integral values. These can be encoded by suitably restricting the codomain of the weight function of octagonal graphs.

Definition 6.5. (Integer octagonal graph.) An *integer octagonal graph* is an octagonal graph $G = (\mathcal{N}, w)$ having an integral weight function:

$$\forall i, j \in \mathcal{N} : w(i, j) \in \mathbb{Z} \cup \{+\infty\}.$$

As an integer octagonal graph is also a rational octagonal graph, the constraint system that it encodes will be satisfiable when interpreted to take values in $\mathbb{Q}$. However, when interpreted to take values in $\mathbb{Z}$, this system may be unsatisfiable since the arcs encoding unary constraints can have an odd weight; we say that an octagonal graph is $\mathbb{Z}$*-consistent* if its encoded integer constraint system is satisfiable. For the same reason, the strong closure of an integer octagonal graph does not provide a canonical form for the integer constraint system.

Definition 6.6. (Graph tight closure/reduction.) An octagonal graph $G = (\mathcal{N}, w)$ is *tightly closed* if it is a strongly closed integer octagonal graph and the following property holds:

$$\forall i \in \mathcal{N} : w(i, \bar{\imath}) \text{ is even.} \tag{7}$$

The *tight closure* of an octagonal graph G in $\mathcal{N}$ is

$$\text{T-closure}(G) := \bigsqcup \bigl\{\, G' \in \mathbb{O} \bigm| G' \trianglelefteq G \text{ and } G' \text{ is tightly closed} \,\bigr\}.$$

An $\mathbb{Z}$-consistent integer octagonal graph R is *tightly reduced* if, for each integer octagonal graph $G \neq R$ such that $R \trianglelefteq G$, we have $\text{T-closure}(R) \neq \text{T-closure}(G)$. A *tight reduction* for the $\mathbb{Z}$-consistent integer octagonal graph G is any tightly reduced graph R such that $\text{T-closure}(R) = \text{T-closure}(G)$.

It follows from these definitions that any tightly closed integer octagonal graph encodes a satisfiable integer constraint system if and only if it is $\mathbb{Z}$-consistent [6, 9]. Therefore, tight closure is a kernel operator on the lattice of octagonal graphs, as was the case for strong closure. Observe also that a tight reduction for a tightly closed graph G is a subgraph of G [9].

To prove the Theorem 6.8 below, we will also use the following result proved in [27, Lemma 4].

Lemma 6.7. *Let $G = (\mathcal{N}, w)$ be an integer octagonal graph with no negative weight cycles and $G_{\mathrm{t}} = (\mathcal{N}, w_{\mathrm{t}})$, where w_{t} satisfies*

$$w_{\mathrm{t}}(i, j) := \begin{cases} 2\lfloor w(i, j)/2 \rfloor, & \text{if } j = \bar{\imath}; \\ w(i, j), & \text{otherwise;} \end{cases}$$

have a negative weight cycle. Then there exist $i, \bar{\imath} \in \mathcal{N}$ and a cycle $\pi = (i \cdot \pi_1 \cdot \bar{\imath}) :: (\bar{\imath} \cdot \pi_2 \cdot i)$ in G such that $w(\pi) = 0$ and the weight of the shortest path in G from i to $\bar{\imath}$ is odd.

Theorem 6.8. *For each $h \in \{1,2\}$, let $\mathrm{oct}_h \in \mathbb{OCT}_n$ be a non-empty integer octagonal shape represented by the tightly closed graph $G_h = (\mathcal{N}, w_h)$ and let R_h be a subgraph of G_h such that* $\text{T-closure}(R_h) = G_h$. *Let also $G_1 \sqcup G_2 = (\mathcal{N}, w)$. Then $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$ if and only if there exists arcs (i,j) of R_1 and (k,ℓ) of R_2 such that, letting $\epsilon_{ij} = 2$ if $j = \bar{\imath}$ and $\epsilon_{ij} = 1$ otherwise and $\epsilon_{k\ell} = 2$ if $\ell = \overline{k}$ and $\epsilon_{k\ell} = 1$ otherwise, the following hold:*

(1a) $w_1(i,j) + \epsilon_{ij} \leq w_2(i,j)$;

(1b) $w_2(k,\ell) + \epsilon_{k\ell} \leq w_1(k,\ell)$;

(2a) $w_1(i,j) + w_2(k,\ell) + \epsilon_{ij} + \epsilon_{k\ell} \leq w(i,\ell) + w(k,j)$;

(2b) $w_1(i,j) + w_2(k,\ell) + \epsilon_{ij} + \epsilon_{k\ell} \leq w(i,\overline{k}) + w(\overline{\ell},j)$;

(3a) $2w_1(i,j) + w_2(k,\ell) + 2\epsilon_{ij} + \epsilon_{k\ell} \leq w(i,\ell) + w(k,\bar{\imath}) + w(\bar{\jmath},j)$;

(3b) $2w_1(i,j) + w_2(k,\ell) + 2\epsilon_{ij} + \epsilon_{k\ell} \leq w(k,j) + w(\bar{\jmath},\ell) + w(i,\bar{\imath})$;

(4a) $w_1(i,j) + 2w_2(k,\ell) + \epsilon_{ij} + 2\epsilon_{k\ell} \leq w(k,j) + w(i,\overline{k}) + w(\overline{\ell},\ell)$;

(4b) $w_1(i,j) + 2w_2(k,\ell) + \epsilon_{ij} + 2\epsilon_{k\ell} \leq w(i,\ell) + w(\overline{\ell},j) + w(k,\overline{k})$.

Proof. We will use the notation $\tilde{\pi}$ as defined in the proof of Theorem 6.3. Suppose that $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$, so that there exists $\mathbf{p} \in \mathrm{oct}_1 \uplus \mathrm{oct}_2$ such that $\mathbf{p} \notin \mathrm{oct}_1$ and $\mathbf{p} \notin \mathrm{oct}_2$. Hence, letting $\tilde{p}_{ij} := \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p})$ and $\tilde{p}_{k\ell} := \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p})$, there exist arcs (i,j) and (k,ℓ) of R_1 and R_2, respectively, satisfying $\tilde{p}_{ij} > w_1(i,j)$ and $\tilde{p}_{k\ell} > w_2(k,\ell)$; as $\mathbf{p} \in \mathrm{oct}_1 \uplus \mathrm{oct}_2$, we also have $w_2(i,j) \geq \tilde{p}_{ij}$ and $w_1(k,\ell) \geq \tilde{p}_{k\ell}$. Note that $w_1(i,j)$ and $w_2(k,\ell)$ are both finite and hence in $\mathbb{Z}$ so that $\tilde{p}_{ij} \geq w_1(i,j)+1$ and $\tilde{p}_{k\ell} \geq w_2(k,\ell)+1$; also, by the tight coherence rule (7), if $j = \bar{\imath}$, $\tilde{p}_{ij} \geq w_1(i,j) + 2$ and, if $k = \overline{\ell}$, $\tilde{p}_{k\ell} \geq w_2(k,\ell) + 2$. Therefore, by definition of ϵ_{ij} and $\epsilon_{k\ell}$, we have

$$\begin{aligned}
w_2(i,j) &\geq \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) \\
&\geq w_1(i,j) + \epsilon_{ij}, \\
w_1(k,\ell) &\geq \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) \\
&\geq w_2(k,\ell) + \epsilon_{k\ell}
\end{aligned}$$

so that conditions (1a) and (1b) hold. Moreover,

$$\begin{aligned}
w(i,\ell) + w(k,j) &\geq \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) \\
&= \tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p}) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) \\
&\geq w_1(i,j) + w_2(k,\ell) + \epsilon_{ij} + \epsilon_{k\ell}
\end{aligned}$$

so that condition (2a) holds and, by a symmetric argument, condition (2b) holds. Similarly,

$$\begin{aligned}
w(i,\ell) + w(k,\bar{\imath}) + w(\bar{\jmath},j) &\geq \bigl(\tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p})\bigr) + \bigl(\tilde{\pi}_k(\mathbf{p}) + \tilde{\pi}_i(\mathbf{p})\bigr) + \bigl(-2\,\tilde{\pi}_j(\mathbf{p})\bigr) \\
&= 2\bigl(\tilde{\pi}_i(\mathbf{p}) - \tilde{\pi}_j(\mathbf{p})\bigr) + \tilde{\pi}_k(\mathbf{p}) - \tilde{\pi}_\ell(\mathbf{p}) \\
&\geq 2w_1(i,j) + w_2(k,\ell) + 2\epsilon_{ij} + \epsilon_{k\ell}
\end{aligned}$$

so that condition (3a) holds; conditions (3b), (4a) and (4b) follow by a symmetrical argument.

We now suppose that, for some $i, j, k, \ell \in \mathcal{N}$, conditions (1a) – (4b) hold. Consider the graph $G' = (\mathcal{N}, w')$ where, for each $r, s \in \mathcal{N}$,

$$w'(r,s) := \begin{cases} -w_1(i,j) - \epsilon_{ij}, & \text{if } (r,s) \in \bigl\{(j,i), (\bar{\imath}, \bar{\jmath})\bigr\}; \\ -w_2(k,\ell) - \epsilon_{k\ell}, & \text{if } (r,s) \in \bigl\{(\ell,k), (\overline{k}, \overline{\ell})\bigr\}; \\ w(r,s), & \text{otherwise.} \end{cases}$$

Let $G := G_1 \sqcup G_2$; as G is coherent, G' is coherent too; as G is tightly closed, G' satisfies property (7). Hence it follows from Lemma 6.7 that G' is $\mathbb{Z}$-consistent if it has no negative weight cycles. By using a reasoning similar to that in the proof of Theorem 6.3, it can be seen that there are no negative weight cycles in G' so that G' is $\mathbb{Z}$-consistent and $G' \trianglelefteq G_1 \sqcup G_2$. Let $\mathrm{oct}' \in \mathbb{OCT}_n$ be represented by G', so that $\varnothing \neq \mathrm{oct}' \subseteq \mathrm{oct}_1 \uplus \mathrm{oct}_2$. Since $w'(j,i) + w_1(i,j) < 0$, we obtain $\mathrm{oct}' \cap \mathrm{oct}_1 = \varnothing$; since $w'(\ell,k) + w_2(k,\ell) < 0$, we obtain $\mathrm{oct}' \cap \mathrm{oct}_2 = \varnothing$. Hence, $\mathrm{oct}_1 \uplus \mathrm{oct}_2 \neq \mathrm{oct}_1 \cup \mathrm{oct}_2$. $\square$

Since the tight closure and tight reduction procedures are both in $\mathrm{O}(n^3)$ [9], the exact join detection algorithm for integer octagonal shapes has the same asymptotic worst-case complexity of all the corresponding algorithms for the other weakly relational shapes.

7. Conclusion and Future Work

Several applications dealing with the synthesis, analysis, verification and optimization of hardware and software systems make use of numerical abstractions. These are sets of geometrical objects —with the structure of a bounded join-semilattice— that are used to approximate the numerical quantities occurring in such systems. In order to improve the precision of the approximation, sets of such objects are often considered and, to limit redundancy and its negative effects, it is important to "merge" objects whose lattice-theoretic join corresponds to their set-theoretic union.

For a wide range of numerical abstractions, we have presented results that state the necessity and sufficiency of relatively simple conditions for the equivalence between join and union. These conditions immediately suggest algorithms that solve the corresponding decision problem. For the case of convex polyhedra, we improve upon one of the algorithms presented in [12, 13] by defining an algorithm with better worst-case complexity. For all the other considered numerical abstractions, we believe the present paper is breaking new ground. In particular, for the case of NNC convex polyhedra, we show that dealing with non-closedness brings significant extra complications. For the other abstractions, the algorithms we propose have worst-case complexities that, in a sense, *match* the complexity of the abstraction, something that cannot be obtained, e.g., by applying an algorithm for general convex polyhedra to octagonal shapes.

All the above mentioned algorithms have been implemented in the Parma Polyhedra Library [7].[8] Besides being made directly available to the client applications, they are used internally in order to implement widening operators over powerset domains [4]. Our preliminary experimental evaluation, though not extensive, showed the efficiency of the algorithms is good, also thanks to a careful coding following the *"first fail" principle*.[9]

Even though preliminary experimentation suggests that —in practice, at least for some applications [4, 16]— pairwise joins allow the removal of most redundancies, work is still needed in the definition of efficient algorithms to decide the exactness of join for $k > 2$ objects. Moreover, it would be useful to develop heuristics to mitigate the combinatorial explosion when attempting full redundancy removal from a set of m objects, as it is clearly impractical to invoke $2^m - m - 1$ times the decision algorithm on $k = 2, \ldots, m$ objects.

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[8] The Parma Polyhedra Library is free software distributed under the terms of the GNU General Public License. See `http://www.cs.unipr.it/ppl/` for further details.

[9] This is a heuristics whereby, in the implementation of a predicate whose success depends on the success of many tests, those that are most likely to fail are tried first.

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