Sameness between based universal algebras (transformations for Menger systems and analytic monoids) *†

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Abstract

The abstract representation-free Algebra of the past century is unable to formalize algebra sameness. A counterexample shows a universal transformation that cannot be representation-free. Then, to define transformations between based universal algebras we must introduce the representations corresponding to the bases, contrary to what was possible in general vector spaces and believed possible in universal algebras.

Our universal notion of transformation comes from a triple characterization concerning three representation facets: the determinations of the *Menger system, analytic monoid* and *endomorphism representation* corresponding to a base. Hence, this notion consists of three equivalent definitions. It characterizes another technical variant and also the universal version of the very semi-linear transformations *that were coordinate-free.*

Universal transformations allow us to check the *actual* invariance of general algebraic constructions, contrary to the seeming invariance of representation-free thinking. Contrary to present beliefs, even the foundation of abstract Linear Algebra turns out to be incomplete.

0 Preliminaries

0.0 Why. According to a widespread view, Linear and Universal Algebra are fairly disjoint fields. They already differ when they define algebra sameness: either by semi-linear transformations or by isomorphisms, up to some

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generalization. (Thus, often [0] vector spaces become couples of a group and of a field, not single universal algebras.) Isomorphisms conveniently define sameness between universal algebras, when the objects of interest are terms, as in Logic and some Computer Science applications [2]. Yet, not all such applications focus on terms, e.g. the ones in **0.4** of [18].

In Linear Algebra, the transformations between frames of reference serve both the theory and the applications. We consider such transformations in order to find some invariant of a theoretical interest. We also perform such transformations merely in order to set a practical problem and to compute its results more easily.

Such transformation do not always concern the same space. We often transform the linear operators of one space into the ones of another, more usually into the usual matrices of the space of the n–tuples over a field. One could not use canonical representations in computations without transformations.

In Universal Algebra, on the contrary, no notion of a transformation appeared. Until a few decades ago, even the elementary notions of a matrix and a representation lacked any universal counterpart. Still, the wider variety occurring in the universal case hints that transformations concern Universal Mathematics, even more than Linear Algebra.

For theoretical purposes, just consider an elementary example. Take the notion of the dimension of a space. A vector space just has a single dimension that clearly is invariant: no need to undergo any transformation of reference frames.

On the contrary, a universal algebra can have bases, namely universal "reference frames", with different numbers of elements. In such a circumstance, we cannot take any of such numbers as a "space" dimension, unless every transformation preserves it. We have to check whether such a strange space is dimensionless or has (many) dimensions by such universal transformations. Without universal transformations, even the dimension notion is blur.

For practical purposes, universal transformations are again necessary. For instance, the ability to convert new equational specifications into free algebras, namely into new data structures, is great. Yet, actual computations on such data require rules for designing convenient canonical representations and changing them according to the problems.

0.1 The problem. Marczewski's weak (or general [2]) isomorphisms formalize a fairly general idea of sameness between two algebras: they match the elementary functions (term operations in ibid.) on the operations rather than the operations themselves. They also are representation-free, like all notions of present abstract Algebra and Universal Algebra. In fact, such isomorphisms are bijections between the algebra carriers, which disregard the representations of endomorphisms that depend on their bases.

Unfortunately, the latter feature prevents to use such an idea to define universal transformations. A counterexample in **3.6** will show that *a single bijection between carriers can have two different transformations depending on two representations:* the former sends any doubleton base onto a different base, whereas the latter does onto the same base. These two ways of "being the same" do not preserve the same algebraic properties. Then, no representation-free notion can formalize sameness, since it cannot identify the transformations.

Outside vector spaces, this nearly states the opposite of what was known within them. As recalled in **0.7**, the generalization from linear transformations (isomorphisms) to the semi-linear ones rose from counterexamples that, conversely, exhibit a single base transition that corresponds to two carrier bijections.

More precisely, we can consider three generalization steps for transformations: linear – semi-linear – universal. Corresponding to them, three identification keys rise: base transition *or* bijection – bijection only – base transition *and* bijection. The transitions in the first key merely concern bases with the same number of reference vectors, while the bijections that can replace them are the corresponding isomorphisms. In the second key, base transitions, unable to identify transformations, drop out, while the bijections can be different from isomorphisms (see **0.7**).

Our problem is in the last key: we have to allow both general base transitions between bases with any cardinality and general bijections without isomorphism restrictions. Vector spaces reach the latter generalization through the semi-linearity condition that also concerns another bijection, not between the carriers: a hint to define universal transformations too by some condition *relating two component bijections.*

Although the problem requires representation dependent tools, not all of them are completely new. The following one has already entered Universal Algebra, yet through a representation-free restricted usage, a restriction we will remove.

0.2 Menger('s) systems. Universal square matrices, recalled in **0.4**, occur in the applications [18] (often before knowing their based algebra, see **0.4** ibid.) for the same reason usual matrices do: they represent endomorphisms.

Linear Algebra is able to exploit endomorphisms through such a representation, namely without the original abstract vector space. It introduces two closely related operations on matrices: the heterogeneous product of a vector times a matrix and the product of two matrices.

The former represents the application of an endomorphism to an element, the latter the functional composition of two endomorphisms. If we see the former as an algebra where the vector identifies an operation taking the matrix columns as its arguments, then we get another presentation of our space.

This is not a peculiarity of vector spaces. As shown in section **6** of [13] or section **1** of [15], Menger's generalization (preceded by [10]) of the former product into superassociative systems with selectors works everywhere after a minor further generalization, our Menger systems as **0.4**, **1.2** and **1.3** will recall. Given any free algebra and one of its bases, there always is such a system, *the Menger system for* this algebra *and* base. This makes our systems different from the "Menger systems" of Universal Algebra (e.g. see [2, 3, 23]), which do not depend on algebra bases.

Our Menger systems, as representations of endomorphism application, again are general presentations of (based) algebras as **1.4** will recall. The Menger system for a based algebra identifies an algebra with the *same carrier* and endomorphism monoid, not one that takes elementary functions ("term operations") from the former algebra as elements. Its operations are all the elementary functions of the former with arguments indexed by the base, as shown in **6.3** and **6.7** of [13].

The superassociative systems of K. Menger differ from our Menger systems only in two features. The former is that Menger's systems might lack selectors, whereas our Menger ones always have them. The latter is the number of such selectors: K. Menger only allowed a finite number, whereas we cannot require it in our universal context. They have to be the elements of a base set of *any* free algebra, see **6.7** of [13].

So far, within our range of applications there are no concrete instances of total superassociative systems without selectors. Besides, any our Menger system χ uniquely defines its selectors (see ibid.). This allows us to forget about mentioning them within our context.

0.3 Notation. We give up any efficient functional notation for the one of Calculus, where repeated functional applications alternate subscripting and right parenthesizing. In spite of this choice of conventional notation, the foundation chosen here is the *pure* set–theoretical one not the conventional algebraic one. Some flaws of the latter (see **0.6** in [18]) concern our treatment.

Hence, we conform to [12], but for the following few differences. We denote the set-theoretical pair $\{\{a\},\{a,b\}\}$ by $\langle a,b\rangle$, yet we still simplify $f(\langle a,b\rangle)$ into $f(a,b)$ and $\langle\langle x,y\rangle,z\rangle$ into (x,y,x) as in [12]. PX denotes the set of subsets of set X and $\boldsymbol{i}_X$ its identity function.

We also cannot follow [12] as far as functional composition is concerned, because of the dangers shown in [16]. We merely consider it as the restriction of relational composition, here denoted by $\cdot$, namely $f \cdot g$ is "the composition of g and f" and $(f \cdot g)(x) = f(g(x))$. Accordingly, we perform the restriction of a function f to some set S merely by functional composition: $f \cdot \boldsymbol{i}_S$.

As usual, we write $f : A \to B$ to say that f is a function with arguments in the whole set A and values in B, $f : A \rightarrowtail B$ or $f : A \twoheadrightarrow B$ to say that it also is one to one or onto B and $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ to say it is a bijection onto B. Yet, as motivated in **0.6** of [18], we will forget that "function – domain" and "family – index" are pairwise synonymic and we avoid the notation $\{a_i\}_{i \in I}$ or $(a_i \mid i \in I)$. Within informal comments we will replace "function" with "labelling", to emphasize arguments, and with "indexing", to emphasize values. Also, we denote the set-theoretical power ${}^{A}B = \{f \mid f : A \to B\}$ as the arithmetic one B^A. (The latter will not occur here.)

0.4 Analytic representations by universal matrices. Although most universal matrices appeared before the acknowledgement of their universal algebras, we recall their theoretical descent from the latter. Such a link clarifies the rôle of their monoids and Menger systems in Universal Mathematics.

Let $\mathbb{E}_\alpha \subseteq A^A$ be the set of all endomorphisms of an algebra α on A. Given a set X, let $b : X \to A$ and consider the function $\boldsymbol{r}_b : \mathbb{E}_\alpha \to A^X$, defined by $\boldsymbol{r}_b(h) = h \cdot b$, for $h \in \mathbb{E}_\alpha$, namely $\boldsymbol{r}_b$ "samples each h at" b by providing each $x \in X$ with the value $h(b(x))$. When a function $b : X \to A$ serves to define such a sampling of endomorphisms, we call it a *frame of* α.

If this sampling serves to represent every endomorphism by any sample and conversely, namely if we got that

$$\boldsymbol{r}_b : \mathbb{E}_\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow A^X \quad , \tag{0}$$

then every structure on $\mathbb{E}_\alpha$ defines another on A^X and we say that

- $\boldsymbol{r}_b$ is an *analytic representation of* $\mathbb{E}_\alpha$, while X is its *dimension set* and the cardinality of X is its *dimension,*
- A^X is the set of the *(square universal) matrices of* α *with respect to* b, while every value $M(x)$ of a matrix $M : X \to A$ is its *column at* $x \in X$,
- b is a *base* or *(universal) reference frame* of α, while its values $b(x)$ are *reference elements* or *selectors* that form the *base set* $B \subseteq A$ for $b : X \twoheadrightarrow B$,
- the $\boldsymbol{r}_b$–image $\circ : A^X \times A^X \to A^X$ of functional composition on $\mathbb{E}_\alpha \subseteq A^A$ is its *matrix product* (that clearly has b as unit),
- b and the function $\chi : A \to A^{A^X}$, defined by (0) from the functional application of endomorphisms, as $\chi_a(\boldsymbol{r}_b(h)) = h(a)$ for $h \in \mathbb{E}_\alpha \subseteq A^A$ and $a \in A$, form the *Menger system derived from* α, with respect to the *frame of selectors* b, of dimension set X and that

- $\circ$ and b form the *monoid of the matrices of* α under $\boldsymbol{r}_b$ or with respect to b, which is isomorphic onto the endomorphism one by definition.

Notice that $A = \emptyset$ by (0) implies $X = \emptyset$, whereas for a singleton A every set X satisfies (0). In the former case we say that *the carrier (of the algebra) is trivial;* in the latter that *the algebra is trivial.* When the algebra is not trivial, $X = \emptyset$ iff $\mathbb{E}_\alpha = \{\boldsymbol{i}_A\}$. It does when all algebra elements are constants. This also implies that

$$X = \emptyset \ \text{ iff } \ \circ : 1 \times 1 \twoheadrightarrow 1 \ \text{ or iff } \ \chi_a(M) = a \text{ for all } M : X \to A \ , \ a \in A. \quad (1)$$

For a nontrivial example, take A as the set of the usual n–tuples of elements of a field and consider any endomorphism of their vector space on the same field. If the reference elements are the ones forming the Kronecker matrix, then their endomorphic images $h(b_x)$ are the column vectors of the usual matrix identifying this endomorphism, the above matrix product turns out to be the familiar one "rows times columns", e.g. as in §2 of ch. IV of [9] (with square matrices), and $\chi_a(M)$ is the corresponding product of vector a times the matrix M.

On the contrary, when such a representation of endomorphisms of a based vector space concerns an arbitrary A and/or an arbitrary base, it gets different matrices and/or different products. Since this is less familiar than the former case, we will disregard it as an example for vector spaces and we will call *usual* the former vector space, as well as its corresponding structures. Yet, we stress that even in vector spaces to be a (universal) matrix is not to be a two-dimensional array, as vectors need not to be one-dimensional arrays.

0.5 Abstract bases. Our universal definition of a base comes from the one of an analytic representation. It identifies the function $\boldsymbol{r}_b$ to state the bijection in (0). On the contrary, the abstract definition of a base as a "free generating family" ignores $\boldsymbol{r}_b$ and considers the two unrelated properties of freedom over itself (or Marczewski's independence [11]) and generability. Still, **6.2**, **6.3** and **6.7** of [13] show that the two definitions are fully equivalent. (Actually, the proof concerns a base such that X is a set of unknowns, yet it trivially extends to any equipotent set X' by a bijection $\ell : X' \twoheadrightarrow X$.) Anyway, when we want to remark our base presentation, we use the synonym "reference frame" from Linear Algebra.

This synonym, as well as others in the above definitions, also serves to recall our technical choice between our "bases as functions" and the "bases as sets", e.g. as in [5, 6]. As known, also this choice does not matter. When we consider a set $X \subseteq A$, instead of a function $b : X \to A$, we still have both (0) and its equivalence to the other definition for most purposes, provided only that we redefine $\boldsymbol{r}_b$ as the restriction function $\boldsymbol{r}_X$ such that $\boldsymbol{r}_X(h) = h \cdot \boldsymbol{i}_X$, for $h \in \mathbb{E}_\alpha$. (Yet, trivial algebras will lack representation dimensions bigger than one, in spite of the fact that they enjoy a lot of other properties.)

Condition (0) restricts both the subsets $E \subseteq A^A$ that can be the sets of all endomorphisms of some algebra α, $E = \mathbb{E}_\alpha$, and the sets A^X of all matrices of some α on A. It also serves to characterize both of them (see **6.8** (D) of [13], **1.3** and **1.4** of [15]), after merely adding the condition that E carries a (sub)monoid of functional composition. Hence, it solves the old problem of characterizing the concrete endomorphism monoid of free algebras. While conventional abstract Universal Algebra did not find any solution (see [1]), this representation dependent solution is almost trivial.

0.6 The solution, this paper presents for our transformation problem, concerns the class of based algebras, where the abstract treatments, mentioned in **0.1** fail. Hence, it also concerns all free algebras, but for a new interpretation of them as shown in **6.6**.

(For the class of all universal algebras this merely is a negative hint: as free algebras are algebras, Abstract Algebra cannot define sameness. Some affirmative hints might come from providing general endomorphism monoids with concrete characterizations, a yet unsolved problem [1, 6].)

We consider three structures coming from the choice of a base. Each of them has its general definition of transformation corresponding to the structure purposes. Yet, the characterization in **6.0** will show the equivalence of the three definitions. Each of them also shares with the semi-linear transformations, we know from general vector spaces, the splitting into two component bijections, one of which is between carriers.

The first structure is the Menger system, we mentioned in **0.2** and **0.4**, which can also rise without a "parent algebra" as **1.2** and **1.3** will explain. Being a standard presentation of based algebras, it defines transformations in **3.3** by two simple properties: a weak preservation of operations and a mutual definability between components.

Here the other component is a bijection between matrices. Contrary to the conventional matrix transformations that operate columnwise, it has a more general functional definition that comes from our more general preservation property. It works on the counterexample **3.6** (A), we mentioned, whereas conventional transformations cannot.

As for vector spaces, the carrier bijections within the same algebra form a group. Yet, contrary to vector spaces, that counterexample also shows in **3.7** that this group is not relevant to the study of invariance, whereas a category is.

The second structure is the monoid of the matrices of **0.4**. It is something more than a monoid: it is an *analytic monoid,* a more general structure, introduced in section **1**, that also has a “dimension set” and it is defined via a basic “combinator”. (Usual monoids just are analytic monoids with a singleton dimension [19].)

This combinator (functional operator) is the one for generating constants. It is a key ingredient of most combinatory systems, including the standard one [7] that employs two combinators only. As such, it is “half of all effective mathematics”. Yet, abstract Algebra ignores it and section **1** has to introduce some properties of its set-theoretical implementations. Section **2** uses it first to define scalars, together with flocks and dilatations, for any based universal algebra and then to characterize the monoid formed by such scalars through a centralizer construction.

Then, to define universal transformations for analytic monoids in **4.5**, we only require to preserve the units and a reduced monoid composition involving this combinator. A first property of such transformations is the preservation of scalar monoids. It universalizes the preservation of scalar fields that semi-linear transformations assume by definition.

The third structure is the representation of the endomorphism monoid. As it contains the one of the dilatations, its universal transformations in **5.2** require both a preservation of endomorphisms and a *full preservation* of universal dilatations. This means that also the “amounts” of dilatations, which come from elements called their indicators, are preserved. Clearly, even semi-linear transformations did require this, but for the formulation, because their dilatations were algebra operations preserved by the field isomorphism.

From the proof that these three universal transformations are the same we get two immediate consequences: the preservation of universal flocks and the characterization of the “representation-free” universal transformations, called *renamings,* that transform matrices columnwise. Further consequences concern another notion of universal transformation that satisfies a condition formally equal to the one for semi-linear transformations: it is equivalent to the former three and generalizes the semi-linear transformations of vector spaces.

0.7 Semi-linear transformations provide vector spaces with a general sameness notion that differs from the abstract one of an isomorphism. (**6.5** (A) will recall their technical details). Isomorphisms (linear transformations) are able to formalize sameness only in a proper subclass of such spaces, corresponding to certain underlying fields, as the real, rational and some Galois ones.

With one of such fields we can identify the transformations that formalize sameness either by base transitions or by carrier bijections (the isomorphisms), since the former determine the latter and conversely. With other fields, as the complex one recalled in **3.5**, also some bijections that are not isomorphisms for vector spaces work as transformations, provided that they are coupled with some field auto(/iso)morphism.

Then, base transitions cannot identify transformations anymore. One transition can have two transformations: this transformation couple and its induced isomorphism, which again corresponds to another couple with the identity as field isomorphism. Such couples, called *semi-linear transformations,* replace isomorphisms when comparing general vector spaces.

This failure of isomorphisms did not weaken the abstract approach of the past century both in Linear Algebra and Universal Algebra. It merely fuelled the idea that vector spaces are fairly peculiar cases of universal algebras, so that one might split their two theories. The "generalized conception of space" and the "uniform method" of A.N. Whitehead (preface of [22]) seemed naive wishes.

In fact, it turned out that even such general carrier bijections were some abstract isomorphisms (between such remarkable algebras as Abelian groups) and that no reference frame was necessary. Moreover, in the universal case, the general isomorphisms (that Marczewski's caution called weak) generalized semi-linear transformations [2], albeit not formally.

On the contrary in **6.4**, one of the minor characterizations of our transformations *formally* generalizes semi-linear transformations to *any based universal algebra.* It overcomes the failure of abstract notions that we will prove in **3.6** (B), because its universal scalars reveal the representation dependence that they hide in vector spaces, as shown in **2.8**.

Then, Whitehead was not so naive. (Also, his treatment of Linear Algebra in [22] was representation dependent.) This also hints that some other abstract beliefs and notions that appear sound and crystal clear might deserve some check.

For instance, we can safely believe that our semi-linear transformations, on which the "first fundamental theorem of projective geometry" [0] relies, are the most general ones for vector spaces, because no other transformation has been found. Yet, we do not have a formal proof. As **6.5** (A) will show, we need some statements that Linear Algebra failed to state and prove.

1 Analytic monoids and constant generators

1.0 Definitions. Let X and A be two sets. Possibly, X can be a natural number $n = \{0, \ldots, n-1\}$. Among the functions in A^X we consider the constant ones. For $a \in A \neq \emptyset$ we denote the one with value a by $\boldsymbol{k}_a$:

$$\boldsymbol{k}_a(x) = a \quad , \tag{2}$$

for all $x \in X \neq \emptyset$. Also, this always defines a *constant generating function* $\boldsymbol{k}: A \to A^X$. In fact, for $X = \emptyset$ and $A \neq \emptyset$ there only are the trivial cases $\boldsymbol{k}_a = \emptyset$ and for $A = \emptyset$ the case $\boldsymbol{k} = \emptyset$.

On A^X consider a binary operation $\circ : A^X \times A^X \to A^X$ (with infix notation) and assume it has a "right $\boldsymbol{K}$–preserved unit", viz. a function $U : X \to A$ with

$$M \circ \boldsymbol{k}_{U(x)} \quad = \quad \boldsymbol{k}_{M(x)} \tag{3}$$

for all $M : X \to A$ and $x \in X$, that also is a "$\boldsymbol{K}$–restricted left unit", viz.

$$U \circ \boldsymbol{k}_a \quad = \quad \boldsymbol{k}_a \tag{4}$$

for all $a \in A$, and satisfies a "$\boldsymbol{K}$–restricted associativity",

$$(M \circ L) \circ \boldsymbol{k}_a \quad = \quad M \circ (L \circ \boldsymbol{k}_a) \ , \tag{5}$$

for all $L, M : X \to A$ and all $a \in A$. Then, we will say that $\circ$ and U define an *analytic monoid of dimension set* X *on* A with the carrier A^X and that U is its *unit.* As shown in **2.1** of [18], (3), the *dimensionality* axiom, generalizes the idea that a Kronecker delta is diagonal, namely that each reference vector lies in its axis.

The requirement that $U : X \to A$ implies that for an empty A one cannot have an analytic monoid, unless X too is empty. In the latter case, the carrier is singleton, whatever A may be, and it also is iff A is, whatever X may be. On the contrary, when the carrier has at least two elements, we – as usual – will say that the analytic monoid is *non trivial.*

Notice that, as far as such set–theoretical cases are concerned, only the first, $A = X = \emptyset$, is completely trivial and only it will allow us to skip definitions and proofs concerning the corresponding analytic monoids: most trivial analytic monoids are not trivial set–theoretically. In fact, even the null or empty dimension case, $X = \emptyset$, determines a single analytic monoid, the trivial one with carrier $1 = \{\emptyset\}$, that *is on every* set A, since $A^\emptyset = \{\emptyset\}$ whatever A is.

Notice also that our three defining conditions are not the three *equational* conditions for monoids, and that (3) involves the dimension set. The first and last of the following properties motivate the name "analytic monoid", which still denotes a mathematical structure different from abstract monoids. (See [19] for details.)

1.1 Recalled properties. We recall that

(Monoid) $\circ$ *and* U *form a monoid on* A^X (proved in **1.7** of [15]);

(χ–definability) $M \circ \boldsymbol{k}_a = \boldsymbol{k}_{(M \circ \boldsymbol{k}_a)(y)}$,
for all $M : X \to A$, $a \in A$ *and* $y \in X$ (proved in **1.7** ibid.);

(Localization) $(M \circ L)(x) = (M \circ \boldsymbol{k}_{L(x)})(x)$,
for all $L, M : X \to A$ *and* $x \in X$ (proved in **2.2** ibid.) and

(Analytic) $\circ$ *and* U *define an analytic monoid on* A *iff they form the monoid of the matrices of some algebra on* A *under the analytic representation* $\boldsymbol{r}_U$ *as in* **0.4** (proved in **1.7** ibid.).

1.2 Definitions. We called the second property χ*–definability,* because it allows us to define a function $\chi : A \to A^{A^X}$, by

$$\chi_a(M) = \begin{cases} a & \text{when } X = \emptyset; \\ (M \circ \boldsymbol{k}_a)(x) & \text{for any } x \in X \neq \emptyset, \end{cases} \tag{6}$$

for all $M : X \to A$ and $a \in A$. This determines an algebra, made of constant–arity operations $\chi_a : A^X \to A$ indexed by the very carrier. We call such an algebra, together with U or without it, the *Menger system derived from* our analytic monoid on A. In fact, **1.5** (C) will show that, given any χ, U is unique. Given $\circ$, if $X \neq \emptyset$, then A and this Menger system are unique. When necessary, we will identify χ as the *algebra of* the Menger system.

In addition to this analytic monoid we will consider another analytic monoid of dimension set Y on B, denoted by $\diamond : B^Y \times B^Y \to B^Y$ and $V : Y \to B$, together with its derived Menger system $\xi : B \to B^{B^Y}$. Hereinafter, we will refer to them as the former and latter monoid respectively. By the Analytic property in **1.1** we can refer to their elements as the former and latter matrices respectively. The same for the derived Menger systems, their elements or operations, their "matrices" of arguments and so on.

Then, from (6) we respectively get

$$L \circ \boldsymbol{k}_a = \boldsymbol{k}_{\chi_a(L)} \text{ , for all } a \in A \text{ and } L : X \to A \text{ and} \tag{7}$$

$$M \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_{\xi_b(M)} \text{ , for all } b \in B \text{ and } M : Y \to B \text{ ,} \tag{8}$$

where $\boldsymbol{k} : A \to A^X$ and $\boldsymbol{\kappa} : B \to B^Y$ respectively denote the former and latter constant generators, both defined as in (2). Hence,

$$\boldsymbol{\kappa}_b(y) = b \text{ for all } b \in B,\ y \in Y \neq \emptyset \text{ ,} \tag{9}$$

while (3) – (5) become

$$M \diamond \boldsymbol{\kappa}_{V(y)} = \boldsymbol{\kappa}_{M(y)} \text{ , for all } y \in Y \text{ and } M : Y \to B, \tag{10}$$

$$V \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b \text{ , for all } b \in B, \text{ and} \tag{11}$$

$$(M \diamond L) \diamond \boldsymbol{\kappa}_b = M \diamond (L \diamond \boldsymbol{\kappa}_b) \text{ , for all } L, M : Y \to B \text{ and } b \in B. \tag{12}$$

Similarly, when we consider two of the Menger systems derived from based algebras, defined in **0.4**, we denote the former and latter analytic representations by

$$\boldsymbol{r}'_U : \mathcal{E} \Vdash\!\!\twoheadrightarrow A^X \quad \text{and} \quad \boldsymbol{r}''_V : \mathcal{F} \Vdash\!\!\twoheadrightarrow B^Y, \tag{13}$$

where $\mathcal{E} \subseteq A^A$ and $\mathcal{F} \subseteq B^B$ respectively denote the set of the endomorphisms of the former algebra and the one of the latter. Therefore, by (0)

$$e \in \mathcal{E} \quad \text{iff} \quad \text{there is } L : X \to A \text{ such that } e(a) = \chi_a(L), \text{ for all } a \in A \tag{14}$$

and

$$f \in \mathcal{F} \quad \text{iff} \quad \text{there is } M : Y \to B \text{ such that } f(b) = \xi_b(M), \text{ for all } b \in B. \tag{15}$$

1.3 Definitions. Given any two functions $U : X \to A$ and $V : Y \to B$, we will also define two *Menger systems,* without deriving them from either an algebra or an analytic monoid, by assigning two functions $\chi : A \to A^{A^X}$ and $\xi : B \to B^{B^Y}$ respectively, which satisfy three conditions each. As this disregards their representation use, often we will call any of them the *algebra of a Menger system.* We will still call U and V the *units* or *frames of selectors.*

The three defining conditions for the former Menger system are:

$$\begin{aligned} \chi_{U(x)}(L) &= L(x) && \text{, for all } L : X \to A \text{ and } x \in X \text{ ;} && (16)\\ \chi_a(U) &= a && \text{, for all } a \in A \text{ and} && (17)\\ \chi_{\chi_a(L)}(M) &= \chi_a(M \circ L) && \text{, for all } a \in A \text{ and } L, M : X \to A \text{ ,} && (18) \end{aligned}$$

where $\circ : A^X \times A^X \to A^X$ here denotes the composition defined by χ in (22). The three for the latter are:

$$\begin{aligned} \xi_{V(y)}(M) &= M(y) && \text{, for all } M : Y \to B \text{ and } y \in Y \text{ ;} && (19)\\ \xi_b(V) &= b && \text{, for all } b \in B \text{ and} && (20)\\ \xi_{\xi_b(M)}(L) &= \xi_b(L \diamond M) && \text{, for all } b \in B \text{ and } L, M : Y \to B \text{ ,} && (21) \end{aligned}$$

where $\diamond : B^Y \times B^Y \to B^Y$ here denotes the composition defined by ξ in (23).

$$\begin{aligned} (M \circ L)_x &= \chi_{L(x)}(M) && \text{, for all } L, M : X \to A \text{ and } x \in X \text{ and} && (22)\\ (M \diamond L)_y &= \xi_{L(y)}(M) && \text{, for all } L, M : Y \to B \text{ and } y \in Y \text{ .} && (23) \end{aligned}$$

The cases $X, A = \emptyset$ are the same as the ones for analytic monoids in **1.0** and (1) continues to hold by (17): when $X = \emptyset$, $\chi : A \Vdash\!\!\twoheadrightarrow A^1$ merely is the generator of singleton constants, while (22) defines $\circ : 1 \times 1 \Vdash\!\!\twoheadrightarrow 1$ trivially.

By the property (Menger to monoid) of **1.4** such compositions together with U or V respectively will define two analytic monoids that we call the *analytic* *monoids derived from* the corresponding Menger systems. The algebras of such systems also define their endomorphism monoids. By the property (Endomorphism) of **1.4** we still denote their carriers by $\mathcal{E}$ and $\mathcal{F}$ respectively, e.g. $\mathcal{E} = \{e : A \to A \mid e(\chi_a(L)) = \chi_a(e \cdot L)$ for all $a \in A$ and $L : X \to A\ \}$.

1.4 Properties. It does not matter how we define analytic monoids and Menger systems nor how they rise, namely

(Algebra to Menger) *the Menger system derived from a based algebra is a Menger system;* conversely,

(Menger to algebra) *every Menger system is derived from an algebra that can be the one of the Menger system, when derived with respect to its unit;*

(Menger to monoid) *the analytic monoid derived from a Menger system is an analytic monoid;*

(Monoid to Menger) *any Menger system derived from an analytic monoid is a Menger system;*

(Monoid loop) *every analytic monoid is derived from the Menger system derived from it;*

(Menger loop) *every Menger system is derived from the analytic monoid derived from it;*

(Endomorphism) *the algebra of the Menger system derived from an algebra keeps its set of endomorphisms.*

Proofs. All such properties are corollaries of **1.4**, **1.7** and **2.2** of [15]. Yet, direct proofs are helpful.

(Algebra to Menger) When $X = \emptyset$ in (0), (16) – (18) are trivially satisfied. Hence, we assume $X \neq \emptyset$. To get (16) start from $(\boldsymbol{r}_U(h))_x = h(U(x))$ for all $h \in \mathcal{E} = \mathbb{E}_\alpha$ and $x \in X$. Then, as defined in **0.4** for $a = U(x)$ and $L = \boldsymbol{r}_U(h) = h \cdot U$, get $\chi_{U(x)}(L) = h(U(x)) = L(x)$, which by (0) holds for all $L : X \to A$ and $x \in X$.

To get (17) start from $\boldsymbol{r}_U(\boldsymbol{i}_A) = \boldsymbol{i}_A \cdot U = U$. Then, as defined in **0.4**, $\chi_a(U) = \boldsymbol{i}_A(a) = a$ for all $a \in A$.

To get (18), we first prove that the operation $\circ : A^X \times A^X \to A^X$, defined in (22), corresponds to the composition of endomorphisms as the one in **0.4** does: $\boldsymbol{r}_U(h \cdot g) = \boldsymbol{r}_U(h) \circ \boldsymbol{r}_U(g)$ for all $g, h \in \mathcal{E} = \mathbb{E}_\alpha$. In fact, for $L = \boldsymbol{r}_U(g)$ and $M = \boldsymbol{r}_U(h)$, $(\boldsymbol{r}_U(h \cdot g))_x = (h \cdot (g \cdot U))_x = (h \cdot \boldsymbol{r}_U(g))_x = h(L(x)) = \chi_{L(x)}(M) = (M \circ L)_x = (\boldsymbol{r}_U(h) \circ \boldsymbol{r}_U(g))_x$ for all $x \in X$. Then, by the definition of derived Menger system in **0.4**, for all $a \in A$, $\chi_{\chi_a(L)}(M) = h(g(a)) = (h \cdot g)_a = \chi_a(M \circ L)$ that by (0) holds for all $L, M : X \to A$.

(Menger to algebra) Assume that χ and U satisfy (16) – (18). According to **0.4** we first show that there is an $\boldsymbol{r}'_U\colon \mathbb{E}_\chi \twoheadrightarrow A^X$, where $\mathbb{E}_\chi = \{e\colon A \to A \mid e(\chi_a(M)) = \chi_a(e\cdot M)$ for all $M: X \to A$ and $a \in A\}$, by providing $\boldsymbol{r}'_U$ with its inverse. Then, such an inverse will extend any sample M onto a χ endomorphism e, its "ηxtension".

Define $\eta: A^X \to A^A$ by $\eta_M(a) = \chi_a(M)$ for all $a \in A$ and $M: X \to A$. Then, for all $e \in \mathbb{E}_\chi$ and $a \in A$ by (17) $((\eta\cdot \boldsymbol{r}'_U)(e))_a = \eta_{e\cdot U}(a) = \chi_a(e\cdot U) = e(\chi_a(U)) = e(a) = (\boldsymbol{i}_{\mathbb{E}_\chi}(e))_a$. Conversely, by (16) $((\boldsymbol{r}'_U\cdot\eta)_M)(x) = (\eta_M\cdot U)(x) = \eta_M(U(x)) = \chi_{U(x)}(M) = M(x) = (\boldsymbol{i}_{A^X}(M))_x$ for all $x \in X \neq \emptyset$ and $M: X \to A$, while $\boldsymbol{r}'_U\cdot\eta = \boldsymbol{i}_{A^X}$ is trivial for $X = \emptyset$.

Now, we only have to check that $\chi_a(\boldsymbol{r}'_U(e)) = e(a)$ for all $e \in \mathbb{E}_\chi$ and $a \in A$, as in **0.4**. This is an extensional rewriting of $\eta\cdot\boldsymbol{r}'_U = \boldsymbol{i}_{\mathbb{E}_\chi}$ we already proved.

(Endomorphism) By the two preceding proofs the Menger system derived from any algebra and the one derived from the algebra of this Menger system coincide. Then, any endomorphism of either algebra is defined by the same χ as in (14). This implies that $\mathbb{E}_\alpha = \mathbb{E}_\chi$.

(Menger to monoid) By **1.1** (Analytic) it is enough to show that the derived monoid is the monoid of the matrices of the algebra of the Menger system. Since the derivation preserves the units, we only have to check the products, which is trivial for $X = \emptyset$. Otherwise, for all $x \in X$, $L = e\cdot U: X \to A$ and $M = f\cdot U: X \to A$ with $e, f \in \mathbb{E}_\chi$, the derived product column (22) equals its corresponding column $((f\cdot e)\cdot U)_x = f(L(x)) = \chi_{L(x)}(M)$ by (Menger to algebra), (Endomorphism) and (14).

(Another possible proof comes from **1.1** (Localization). In fact, in **2.2** of [15] this localization property identifies a class of binary operations, where each of the analytic axioms (3) – (5) is equivalent to the corresponding Menger one in (16) – (18).)

(Monoid to Menger) Again, the case $X = \emptyset$ and the units were already checked. All Menger axioms (16) – (18) come from their corresponding analytic axioms (3) – (5) by (6) and (2) or (7) and the associativity in **1.1** (Monoid): $\chi_{U(x)}(L) = (L\circ\boldsymbol{k}_{U(x)})(x) = \boldsymbol{k}_{L(x)}(x) = L(x)$, $\chi_a(U) = (U\circ\boldsymbol{k}_a)(x) = \boldsymbol{k}_a(x) = a$ and $\chi_{\chi_a(L)}(M) = (M\circ\boldsymbol{k}_{\chi_a(L)})(x) = (M\circ(L\circ\boldsymbol{k}_a))(x) = ((M\circ L)\circ\boldsymbol{k}_a)(x) = \chi_a(M\circ L)$, for all $a \in A$ and $L, M: X \to A$ and any $x \in X$.

(Monoid loop) Let $\circ$ and U be as in **1.0**. Since the derivations in **1.2** and **1.3** keep U, we only check $\circ$. When $X = \emptyset$, $\circ: 1\times 1 \to 1$. Then, by (6) a derived Menger system is any generator of singleton constants $k: A \twoheadrightarrow A^1$ for some A, and (22) returns our $\circ$ trivially. Therefore, we assume $X \neq \emptyset$ and the required equality, $(M\circ L)_x = \chi_{L(x)}(M) = (M\circ\boldsymbol{k}_{L(x)})(x)$ for all $x \in X$ and $L, M: X \to A$, comes from the property **1.1** (Localization).

(Menger loop) Let χ and U be as in (16) – (18). Since the derivations keep

U, we only check χ for $X \neq \emptyset$ (see above for the case $X = \emptyset$). Now, the required equality merely comes from (2): $\chi_a(M) = (M \circ \boldsymbol{k}_a)(x) = \chi_{\boldsymbol{k}_{a(x)}}(M)$ for all $a \in A$, $M : X \to A$ and for some $x \in X \neq \emptyset$. *Q.E.D.*

1.5 Corollaries.

(A) *$\circ$ and U form an analytic monoid iff they define the monoid derived from some Menger system and iff they form the monoid of the matrices of its algebra with respect to its unit.*

(B) *The algebras χ and ξ of two Menger systems derived from the same algebra α, with respect to possibly different reference frames, have the same endomorphisms:*

$$\mathcal{E} = \mathcal{F} \subseteq A^A \ . \tag{24}$$

(C) *The algebra of a Menger system determines its frame of selectors.*

Proofs. (A) The former (iff) comes from (Monoid loop) and (Menger to monoid) in **1.4**. The latter from (Analytic) in **1.1** and (Endomorphism) in **1.4**, because of (0), as in the proof of **1.4** (Menger to monoid).

(B) By the property (Endomorphism) of **1.4** $\mathcal{E} = \mathbb{E}_\alpha = \mathcal{F}$.

(C) From the uniqueness of monoid units through **1.4** (Menger to monoid) and **1.1** (Monoid). *Q.E.D.*

1.6 Definitions. Consider our constant generators $\boldsymbol{k} : A \to A^X$ and $\boldsymbol{\kappa} : B \to B^Y$. When $C \subseteq A^X$ and $D \subseteq B^Y$ denote the two corresponding sets of constant functions, unless $A = X = \emptyset$, we get two bijections,

$$\boldsymbol{k} : A \rightarrowtail\!\!\!\twoheadrightarrow C \ , \text{ when } \ X \neq \emptyset, \ \text{ and } \ \boldsymbol{\kappa} : B \rightarrowtail\!\!\!\twoheadrightarrow D \ , \text{ when } \ Y \neq \emptyset, \tag{25}$$

or two constants: $X, Y = \emptyset$ respectively imply $C, D = 1 = \{\emptyset\}$, $\boldsymbol{k} : A \twoheadrightarrow C$ and $\boldsymbol{\kappa} : B \twoheadrightarrow D$. Also, $C = A^X$ iff A or X is at most singleton. Likewise for D.

We say that a function $t : A^X \to B^Y$ *retypes* $\boldsymbol{K}$, when for all $f : X \to A$ $t(f)$ is constant iff f is. This is the same as to require that $t \cdot \boldsymbol{i}_C$ is onto D. We also say that a bijection $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ for $X, Y \neq \emptyset$ *depicts elements as constants,* when there exists a bijection $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$ such that

$$t \cdot \boldsymbol{k} = \boldsymbol{\kappa} \cdot g \ . \tag{26}$$

(This cannot extend to the cases $X, Y = \emptyset$, where t does not determines A and/nor B, as it should become a property of A and/or B, not of t.)

1.7 Lemmata.

(A) *When $X, Y \neq \emptyset$, a bijection $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ retypes $\boldsymbol{K}$ iff it depicts elements as constants.*

(B) *When a bijection* $t : A^X \twoheadrightarrow B^Y$ *retypes* $\boldsymbol{K}$ *and* A *has at least two elements, if* X *is singleton, then* Y *is.*

Proofs. (A) (Only if) As $t \cdot \boldsymbol{i}_C : C \twoheadrightarrow D$, $g = \boldsymbol{\kappa}^{-1} \cdot t \cdot \boldsymbol{k}$ provides us the required bijection by (25). (If) Since by (26) $t \cdot \boldsymbol{i}_C = t \cdot \boldsymbol{k} \cdot \boldsymbol{k}^{-1} = \boldsymbol{\kappa} \cdot g \cdot \boldsymbol{k}^{-1}$, the function $t \cdot \boldsymbol{i}_C$ is onto D, as required.

(B) When X is singleton, $C = A^X$ and it has at least two elements. Then, $t \cdot \boldsymbol{i}_C = t$ is onto both $B^Y \supseteq D$ and D. This implies that $D = B^Y$ and that it has at least two elements. No B with less than two elements can do it. Hence, Y too is singleton. *Q.E.D.*

1.8 Definition. When $X, Y \neq \emptyset$ and $t : A^X \twoheadrightarrow B^Y$ retypes $\boldsymbol{K}$, we say that t $\boldsymbol{K}$*–induces* the above $g : A \twoheadrightarrow B$. Clearly, (26) defines at most one g. By (9) and (26) the $\boldsymbol{K}$–induced bijection is defined by $g(a) = \boldsymbol{\kappa}_{g(a)}(y) = t(\boldsymbol{k}_a)(y)$, for all $a \in A$ and every $y \in Y$.

2 Flocks and dilatations.

2.0 Definitions. Universal transformation will require to generalize some simple notions that we know from vector spaces to any based universal algebra. We say that $c \in A$ is a *flock combiner of* χ or *of* the Menger system of χ, when

$$\chi_c(\boldsymbol{k}_a) = a \text{ , for all } a \in A \text{ .} \tag{27}$$

Then, the element of a singleton A is a flock combiner. Hence, for $X = \emptyset$ by (1) and (6) $c \in A$ is a flock combiner iff A is singleton. Yet, things are less trivial for nontrivial dimensions.

For instance, when χ is the usual multiplication of a vector times a matrix of **0.4**, a vector $c = [c_0, c_1, c_2]$ satisfies condition (27) defining flock combiners, i.e. it satisfies

$$\begin{aligned} a_0c_0 + a_0c_1 + a_0c_2 &= a_0 \\ a_1c_0 + a_1c_1 + a_1c_2 &= a_1 \\ a_2c_0 + a_2c_1 + a_2c_2 &= a_2 \text{ ,} \end{aligned}$$

for all $a = [a_0, a_1, a_2]^\top$, iff $c_0 + c_1 + c_2 = 1$, namely iff the *linear invariant* of the matrix $\boldsymbol{k}_c$ with all columns equal to c is the unit.

Reference vectors always are flock combiners. Yet, in general (the plane on GF(2) is one of the exceptions) other vectors can be. This exactly is what happens in the universal case: as **2.1** (A) will show, each U_x and each V_y are flock combiners. Yet, they are fairly peculiar, since as shown in **2.1** of [18] any reference element belongs to its axis, as a reference vector does, i.e. all its components but one are null, whereas a general flock combiner might lack null components. (Even in any universal based algebra an element can have null components, see [18].)

As known (e.g., see the lemma in **VII.7** of [0]), in our space of dimension *three* the latter c's define any flock of dimension up to *two* from a triple of vectors, or also *any* flock of a two dimensional projective space. In general, a flock in a vector space can also use flock combiners from vector spaces of a different dimension, e.g. in order to state that all the space is a flock. Moreover, we know that being a flock does not depend from the choice of a reference frame in the former space.

However, we will not need such a generalization for our universal flocks: here they merely serve to compare structures coming from the choice of our two reference frames. Besides, the above vector space flock can become the flock of another triple, after changing the reference frame. Hence, so far universal flock combiners only define a *(universal) flock* $\Phi'_L \subseteq A$ *with respect to* χ by

$$\Phi'_L = \{\chi_c(L) \mid c \text{ is a flock combiner}\} \tag{28}$$

from any matrix $L : X \to A$. When χ is derived from a given algebra, by **1.4** (Menger to algebra) we can say that such a $\chi_c(L)$ is the L*–combination* of flock combiner c *with respect to* U and that Φ'_L is the L*–flock with respect to* U.

When L is our reference frame U, we will also say that flock Φ'_U is the *reference flock of* χ or *with respect to* U; likewise we define the reference flock Φ''_V of ξ. In **2.1** (C) this allows us to see combiners as combinations.

2.1 Corollaries.
(A) *Bases are made of flock combiners,* $U : X \to \Phi'_U$ *and* $V : Y \to \Phi''_V$.

(B) In general, *each column of every matrix is a matrix combination,* $L : X \to \Phi'_L$ *for all* $L : X \to A$.

(C) *The set of all flock combiners is the reference flock.*

(D) *The flocks of non trivial constants are the singletons of their values:* $\Phi'_{\boldsymbol{k}(a)} = \{a\}$ *for all* $a \in A$ *with* $X \neq \emptyset$.

Proofs. (A) In fact, for all $x \in X$, $\chi_{U(x)}(\boldsymbol{k}_a) = \boldsymbol{k}_a(x)$ by (16). Hence, $\chi_{U(x)}(\boldsymbol{k}_a) = a$ by (2). Likewise for V by (19) and (9). The statement is trivial for $X = \emptyset$.

(B) By (A) $c = U(x)$ is a flock combiner for each $x \in X$ and by (16) $\chi_c(L) = L(x)$. Again, $X = \emptyset$ is trivial.

(C) This comes from axiom (17) (or from (20)).

(D) From (28) and (27). *Q.E.D.*

2.2 Definitions. Flock combiners are a case of the *dilatation indicator* defined in the former Menger system as an element $c \in A$ such that $\chi_c{\cdot}\boldsymbol{k} : A \to A$ is any endomorphism $e \in \mathcal{E}$ of χ. In such a case, e and its matrix $S = e \cdot U : X \to A$ are respectively called a *dilatation* and a *(universal) scalar* of χ (see **2.2** of [18], [14] and **5.1** of [13]), while c is called an *indicator of* e or *of* S.

In fact, (27) states that e is the identity on A (which always is in $\mathcal{E}$), namely flock combiners merely are the indicators of the identity. They also are general dilatation indicators up to the dilatations themselves, as the following theorem **2.4** will show.

The above dilatations are not all the ones of a Menger χ. When $X = \emptyset$, we say that $\boldsymbol{i}_A$ and its matrix $S = \emptyset$ are *the dilatation* and *the scalar* of $\chi : A \to A^1$ respectively, even for a non singleton A, namely even when there are not dilatation indicators. This is a split definition, yet it comes from the unsplit one in **2.5** of [14] for general universal algebras.

The latter uses unary elementary functions, not indicators, in order to define a dilatation as an isotropic endomorphism, without any splitting. (Such a unarity formalizes the isotropy condition for endomorphisms that concerns their "geometric" dimensions as in **5.1** of [13].) This does not matter till X has at least one element: any X–ary elementary functions is a χ_c, as mentioned in **0.2**, and we get any dilatation as $\chi_c \cdot \boldsymbol{k}$, for some indicator c.

On the contrary, when $X = \emptyset$, every elementary function χ_a is a nullary constant. Unless A is singleton, no nullary function can replace the identity. Yet, the identity, the only endomorphism, always satisfies the recalled isotropy. Then, when the general definition applies to the algebra of a Menger system, both indicator defined dilatations and (in the last case) an identity without indicators can rise.

Anyway, the characterization in **2.6** (A) of scalars will avoid any splitting, as the recalled definition of general dilatation did. This characterization formally disregards any indicator and any dilatation. It also is fully analytic in the sense that it uses the multiplication of an analytic monoid to state a "$\boldsymbol{K}$-restricted" commutativity.

(One might well consider such a characterization as a standard centralizer construction for an abstract monoid. Yet, this does not enough stress the rôle of combinator $\boldsymbol{K}$ in analytic monoids we mentioned in **0.6**.)

Our split definition introduces scalars by dilatations also in order to show easily that universal scalars do correspond to the scalars we know from vector spaces, as we did in **2.0** for the unit scalar and we will do in **2.8** for the others. As shown in the following, even the properties of indicators are extensions of the ones of flock combiners.

Though indicators are not formally necessary to define scalars, they are useful, as they determine the "amount" of a dilatation by an element, instead of by a matrix, as a scalar does. This will allow *dilatations to relate with* *carrier bijections.* Yet, while a dilatation has a single matrix, in general it has a set of indicators, possibly an empty one. I_e will denote the set of indicators of dilatation e.

$F \subseteq A^X$ and $G \subseteq B^Y$ will respectively denote the sets of scalars of χ and ξ. $\Delta \subseteq \mathcal{E}$ and $\Gamma \subseteq \mathcal{F}$ will respectively denote the corresponding sets of dilatations. By **2.7** (B) and (C) in both cases such sets carry monoids that we respectively call the *scalar monoid* and the *dilatation monoid* of the corresponding reference frames, Menger systems or analytic monoids. Clearly, for $X = \emptyset$ they are fairly trivial, since $F = \{\emptyset\}$ and $\Delta = \{\boldsymbol{i}_A\}$.

2.3 Lemma. *c is a dilatation indicator in the former Menger system iff there exists $L : X \to A$ such that $\chi_c(\boldsymbol{k}_a) = \chi_a(L)$ for all $a \in A$. Likewise in the latter Menger system: d is iff there exists $M : Y \to B$ such that $\xi_d(\boldsymbol{\kappa}_b) = \xi_b(M)$ for all $b \in B$. Such an L and M are the scalars of the corresponding dilatations.*

Proof. The (iff) parts come from (14) and (15), while the scalar observations from (13) by (16) and (19), e.g. $(e \cdot U)(x) = ((\chi_c \cdot \boldsymbol{k}) \cdot U)(x) = \chi_c(\boldsymbol{k}_{U(x)}) = \chi_{U(x)}(L) = L(x)$ for all $x \in X \neq \emptyset$, while for $X = \emptyset$ it is trivial, $L = \emptyset$. *Q.E.D.*

2.4 Theorem. *For every scalar $S : X \to A$ of χ, the value $c = \chi_u(S)$ of its dilatation at any flock combiner $u \in \Phi'_U$ is an indicator of S.*

Proof. By **2.3** we can show that, for all $a \in A$, $\chi_a(S) = \chi_c(\boldsymbol{k}_a) = \chi_{\chi_u(S)}(\boldsymbol{k}_a)$. Since S is a scalar and $\Phi'_U \neq \emptyset$, it has some indicator d, as remarked in **2.2**. Hence, $\chi_d(\boldsymbol{k}_a) = \chi_a(S)$ for any $a \in A$. Therefore, $c = \chi_u(S) = \chi_d(\boldsymbol{k}_u)$ and $\chi_c(\boldsymbol{k}_a) = \chi_{\chi_d(\boldsymbol{k}_u)}(\boldsymbol{k}_a)$.

Now, by the properties (Menger loop) and (Menger to monoid) in **1.4** we can use (18) and (7) to get, for all $a \in A$, that $\chi_c(\boldsymbol{k}_a) = \chi_d(\boldsymbol{k}_a \circ \boldsymbol{k}_u) = \chi_d(\boldsymbol{k}_{\chi_u(\boldsymbol{k}_a)})$. This allows us to exploit (27) for $u \in \Phi'_U$ and **2.3** to get $\chi_c(\boldsymbol{k}_a) = \chi_d(\boldsymbol{k}_a) = \chi_a(S)$ as required. *Q.E.D.*

2.5 Corollary. *For every scalar $S : X \to A$ of χ, each column S_x for $x \in X \neq \emptyset$ is a dilatation indicator of S: for all $e \in \Delta$, $e \cdot U : X \to I_e$.*

Proof. By (16) we get such a column as $S_x = \chi_{U(x)}(S)$ and by **2.1** (A) U_x is a flock combiner. Hence, by the above theorem S_x is an indicator of S for each $x \in X$. *Q.E.D.*

2.6 Theorems.

(A) *A matrix $S : X \to A$ is a scalar of χ iff $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$.*

(B) *The product of a matrix $L : X \to I_e$ of indicators of a dilatation $e \in \Delta$ times one $M : X \to I_f$ for a dilatation $f \in \Delta$ is a matrix $M \circ L : X \to I_{e \cdot f}$ of indicators of the commuted corresponding composition.*

Proofs. (A) (Only if) When $X = \emptyset$, there only is one matrix and the statement is trivial, because either $A = \emptyset$ or $\boldsymbol{k}_a = \emptyset = S$. Otherwise, by (6),

2.5 and **2.3** we get $(S \circ \boldsymbol{k}_a)(x) = \chi_a(S) = \chi_{S(x)}\boldsymbol{k}_a$ for all $a \in A$ and $x \in X$. Then, we can use (16), (18) and (16) again to get $\chi_{S(x)}\boldsymbol{k}_a = \chi_{\chi_{U(x)}(S)}\boldsymbol{k}_a = \chi_{U(x)}(\boldsymbol{k}_a \circ S) = (\boldsymbol{k}_a \circ S)(x)$ for all $a \in A$ and $x \in X$. Hence, $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$.

(If) We can again consider an $X \neq \emptyset$. From the second chain of passages in the (Only if) part we know that, for each $x \in X$ and for all $a \in A$, $(\boldsymbol{k}_a \circ S)(x) = \chi_{S(x)}\boldsymbol{k}_a$. From **1.4** and (6) we also know that $(S \circ \boldsymbol{k}_a)(x) = \chi_a(S)$. Then, when $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$, there is an $x \in X$ such that $\chi_{S(x)}\boldsymbol{k}_a = \chi_a(S)$ for all $a \in A$. Hence, by **2.3** S is a scalar.

(B) When $X = \emptyset$, it follows from (1) as recalled in **1.3**. Otherwise, for every $x \in X$ and $a \in A$ by (22) $(\boldsymbol{k}_a \circ M)_x = \chi_{M(x)}(\boldsymbol{k}_a) = f(a)$, since by **2.5** $M(x) \in I_f$. Namely, by (2) $\boldsymbol{k}_a \circ M = \boldsymbol{k}_{f(a)}$. Then, for every $x \in X$ consider $c = (M \circ L)_x$. By (22) $c = \chi_{L(x)}(M)$ and by (18), (2) and **2.3**, for all $a \in A$, $\chi_c(\boldsymbol{k}_a) = \chi_{L(x)}(\boldsymbol{k}_a \circ M) = \chi_{L(x)}(\boldsymbol{k}_{f(a)}) = e(f(a))$, since by **2.5** $L(x) \in I_e$, namely $c \in I_{e \cdot f}$. *Q.E.D.*

2.7 Corollaries.

(A) *For every scalar $S : X \to A$ of χ, let $c = \chi_u(S)$ be the value of its dilatation at any $u \in A$, then, if c is an indicator of S and the dilatation is one to one, u is a flock combiner, $u \in \Phi'_U$.*

(B) *Scalars form a submonoid of the analytic monoid.*

(C) *Dilatations form a submonoid of the endomorphism monoid and the scalar monoid is the isomorphic image of the dilatation monoid under the analytic representation.*

(D) *The product of matrices of flock combiners is a matrix of flock combiners: $\circ' : \Phi_U^X \times \Phi_U^X \to \Phi_U^X$, where $\circ' = \circ \cdot \boldsymbol{i}_{\Phi_U^X \times \Phi_U^X}$ denotes this restriction of the product.*

Proofs. (A) For each $a \in A$, from the indicator premise $\chi_a(S) = \chi_{\chi_u(S)}(\boldsymbol{k}_a)$ as in **2.3** by (18) we get $\chi_a(S) = \chi_u(\boldsymbol{k}_a \circ S)$. Hence, theorem **2.6** (A) and (18) get $\chi_a(S) = \chi_u(S \circ \boldsymbol{k}_a) = \chi_{\chi_u(\boldsymbol{k}_a)}(S)$ for all $a \in A$. Therefore, by (14) the injectivity premise gets $\chi_u(\boldsymbol{k}_a) = a$ for all $a \in A$, as required by (27).

(B) The unit U is a scalar because of **2.6** (A) and **1.1** (Monoid). Given two scalars S' and S'' as in **2.6** (A), we get $(S' \circ S'') \circ \boldsymbol{k}_a = S' \circ \boldsymbol{k}_a \circ S'' = \boldsymbol{k}_a \circ (S' \circ S'')$ for all $a \in A$.

(C) The analytic representation is one to one, as it was defined in **0.4**, while by definition **2.2** $\boldsymbol{r} \cdot \boldsymbol{i}_\Delta$ is onto F, $\boldsymbol{r} \cdot \boldsymbol{i}_\Delta : \Delta \rightarrowtail\!\!\!\twoheadrightarrow F$, and preserves the units. Then, by (B) we only need to show that $\boldsymbol{r}_U(e'' \cdot e') = \boldsymbol{r}_U(e'') \circ \boldsymbol{r}_U(e')$ for all $e', e'' \in \Delta$, which again comes from **0.4**.

(D) In **2.6** (B) take $e = f = \boldsymbol{i}_A$ and use **2.1** (C). *Q.E.D.*

2.8 Example. Given a non trivial vector space with two reference frames $U, V : X \to A$, let F denote the carrier of its field, namely the set of its "vector-space scalars". In the former analytic monoid, consider the function $D : \mathrm{F} \to A^X$ that provides each number $s \in \mathrm{F}$ with its diagonal matrix $D_s = \overline{s} \cdot U$, where $\overline{s} : A \to A$ is the multiplication by s, namely $D_s(x) = sU_x$ for each $x \in X$. Likewise, in the latter consider $D' : \mathrm{F} \to A^X$ with $D'_s = \overline{s} \cdot V$.

We claim that $D : \mathrm{F} \Vdash\!\!\twoheadrightarrow F$ *is an isomorphism from the monoid of the field product onto the scalar monoid of* U*, which determines dilatations that are invariant with respect to the choice of the analytic representations in* **2.7** (C)*:*

$$\chi_a(D_s) = sa = \xi_a(D'_s) \ , \quad \textit{for all } a \in A \textit{ and } s \in \mathrm{F} \ . \tag{29}$$

Proof. Since $\overline{s} \in \mathcal{E}$, (29) follows from (Endomorphism) in **1.4**, (17) and (20): e.g. $\chi_a(D_s) = \chi_a(\overline{s} \cdot U) = \overline{s}(\chi_a(U)) = \overline{s}(a) = sa$. Clearly, D is one to one and preserves the products. Finally, we prove that $S \in F$ iff $S = D_s$ for some $s \in \mathrm{F}$.

The (if) follows from **2.6** (A), because in a vector space $\overline{s} \cdot e = e \cdot \overline{s}$ for all $e \in \mathcal{E}$, which by **2.7** (C) implies $D_s \circ M = M \circ D_s$ for all $M : X \to A$, hence for all $M = \boldsymbol{k}_a$. The (only if) from **2.5**. By **2.3** it implies that, for each $x \in X$, $\chi_{S(x)}(\boldsymbol{k}_a) = \chi_a(S)$ for all $a \in A$. This states that $\chi_{S(x)} \cdot \boldsymbol{k}$ is linear, because, as mentioned in **0.2**, it is an elementary function of our vector space, and that it does not depend on x. Since it also is unary, it must be the multiplication times a field number, $\chi_{S(x)} \cdot \boldsymbol{k} = \overline{s}$ for some fixed $s \in \mathrm{F}$, which implies $S = \overline{s} \cdot U$. Hence, $D : \mathrm{F} \Vdash\!\!\twoheadrightarrow F$. *Q.E.D.*

In this proof we used the commutativity of the field product. As **2.9** (B) will show, this commutativity is not the minor property one could painlessly get rid of. "Scalars" in a skew field may not be universal scalars for its moduli.

2.9 Theorems.
(A) *The set of indicators* c *of a bijective dilatation* $e = \chi_c \cdot \boldsymbol{k} : A \Vdash\!\!\twoheadrightarrow A$ *with scalar* $S = e \cdot U$ *is the flock of the* S*–combinations:* $I_e = \Phi'_S$.

(B) *The scalar monoid is commutative.*

Proofs. (A) Since e is onto A, we get any indicator as $c = e(u) = \chi_u(S)$ for some $u \in A$. Hence, to prove that Φ'_S is the set of such c's, by (28) and **2.1** (C) we can prove that the reference flock Φ'_U is the set $\mathcal{U}$ of such u's. Since e is one to one, the proof that $\mathcal{U} \subseteq \Phi'_U$ is **2.7** (A), while the proof of $\mathcal{U} \supseteq \Phi'_U$ is **2.4**.

(B) Let $S', S'' \in F$ be the scalars of two dilatations $e, f \in \Delta$ respectively. By **2.5** $S' : X \to I_e$ and $S'' : X \to I_f$. Then, by **2.6** (B) $S'' \circ S' : X \to I_{e \cdot f}$. By **2.7** (B) it also is a scalar, nay by **2.5** and the injectivity of (13) the scalar of $e \cdot f$. Hence, $S'' \circ S' = \boldsymbol{r}'_U(e \cdot f) = S' \circ S''$ by **0.4**. *Q.E.D.*

3 Descriptions.

3.0 Definitions. The first notion of transformation in **3.3** will concern Menger systems. It will use some set-theoretical properties of the following preliminary notions, which also concern the crucial counterexample **3.6**.

Given the two Menger systems of **1.3** and a bijection

$$g : A \rightarrowtail\!\!\!\twoheadrightarrow B \ , \tag{30}$$

consider the relation $t \subseteq A^X \times B^Y$ defined for all $L : X \to A$ and $M : Y \to B$ by $\langle L, M\rangle \in t$ iff for all $a \in A$

$$g(\chi_a(L)) = \xi_{g(a)}(M) \ . \tag{31}$$

An example of such a relation t is the one of a transformation of the matrices of two based vector spaces, where $Y = X$ and $\langle L, M\rangle \in t$ iff

$$M = g \cdot L \tag{32}$$

for a linear or semi-linear transformation $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$, which implies $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$.

Notice that in general, whenever we consider two based algebras deriving our two Menger systems as in **1.4**, the *choice of the bases*, not an a priori assumption like (32), determines t from g. The counterexample in **3.6** (A) will show that (31) does not imply (32) nor its generalization $M = g \cdot L \cdot l^{-1}$ for any $l : X \rightarrowtail\!\!\!\twoheadrightarrow Y$. Besides, in the proof of **3.1** (B) we will see that the mere requirement that (31) holds for certain a's ensures that for each L in the domain of t there only is one way to get M. The universal formula that expresses such one way and replaces (32) is (34).

If t relates every former matrix L with some latter matrix M and t^{-1}, conversely, every M with some L, then we will say that g *totally induces* t and we denote the function relating the g's to the t's by $T \subseteq B^A \times \mathsf{P}(A^X \times B^Y)$. We will also call the function $V' = g^{-1} \cdot V : Y \to A$ the *(algebraically) converse base (with respect to g)*. Here, V' need not to be a converse of V in the sense that $\langle V', V\rangle \in t$. We are merely recalling the restricted notion of t in (32) or (35) that comes from Algebra.

3.1 Lemmata. *If $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$ totally induces t as above, then*

(A) (when one of the sets of matrices is singleton) *trivial dimensions must coexist, $X = \emptyset$ iff $Y = \emptyset$, or both Menger systems have trivial algebras, hence in both cases*

$$A^X = \{U\} \ \text{ iff } \ B^Y = \{V\} \ , \tag{33}$$

(B) *the induced relation is a bijection, $t = T_g : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ and,*

(C) *given* χ, T_g *depends only on* V, *through the converse base* $V' : Y \to A$, *by*

$$(T_g(L))_y = g(\chi_{V'(y)}(L)) \quad \textit{for all } L : X \to A \textit{ and } \ y \in Y \ . \tag{34}$$

Proofs. (A) The coexistence of algebra triviality comes from (30). Then, consider dimension triviality with non trivial algebras. As observed in **1.3**, when $X = \emptyset$, χ is the generator of singleton constants $k : A \Vdash\!\!\twoheadrightarrow A^1$. Then, $g(a) = \xi_{g(a)}(M)$ in (31). Since g is onto B, any M behaves as V in (20). By (23), **1.1** (Monoid) and **1.4** $M = V$, because a left unit of a monoid is its only unit. Hence, the total induction assumption implies $B^Y = \{V\}$. As B is not singleton, this implies $Y = \emptyset$. Conversely, for $Y = \emptyset$ we consider g^{-1}.

((B) and (C)) When A and B are singleton, both A^X and B^Y are. Hence both statements easily follow from (31) and (33). When $X = \emptyset$, by (A) the induced relation is the singleton function $t : 1 \Vdash\!\!\twoheadrightarrow 1$ and (34) holds trivially. Otherwise, we can assume that both $X, Y \neq \emptyset$ and, hence, $A \neq \emptyset$.

Let us show that $t : A^X \to B^Y$. From **3.0**, for all $\langle L, M\rangle \in t$, (31) holds in particular for each $a = V'(y) = g^{-1}(V(y))$ with $y \in Y$. Hence, for all $y \in Y$ by (19) and (30)

$$M(y) = \xi_{V(y)}(M) = \xi_{g(g-1(V(y)))}(M) = \xi_{g(a)}(M) = g(\chi_a(L)) = g(\chi_{V'(y)}(L)) \ .$$

Then, $M = T_g(L)$ as in (34) and $t = T_g$.

Since $t : A^X \twoheadrightarrow B^Y$ comes from the total induction assumption, now we only have to show $t : A^X \Vdash\!\!\to B^Y$. This, easily follows after building the converse of (34). In fact, (31) by (30) becomes its converse: $g^{-1}(\xi_b(M)) = g^{-1}(g(\chi_{g^{-1}(b)}(L))) = \chi_{g^{-1}(b)}(L)$ for all $b = g(a) \in B$. This defines t^{-1}, which is totally induced by g^{-1}, since t was by g. From this converse of (31) we get the converse of (34), by using the converse base $U' : X \to B$ with respect to g^{-1}: for all $M : Y \to B$ and $x \in X$, $(t^{-1}(M))_x = g^{-1}(\xi_{U'(x)}(M))$. This redefines t^{-1} as a function. Hence, t is one to one. *Q.E.D.*

3.2 Corollaries. *When* $g : A \Vdash\!\!\twoheadrightarrow B$ *totally induces* t *as above:*

(A) t *preserves the frames of selectors,* $t(U) = V$ *and,*

(B) *if* $t = T_g : A^X \Vdash\!\!\twoheadrightarrow B^Y$ $\boldsymbol{K}$*–induces* g *as in* **1.8** *or if the two dimensions are trivial* (namely, if t retypes $\boldsymbol{K}$ as in **1.6**), *then* g *preserves reference flocks in both ways:* $c \in \Phi'_U$ *iff* $g(c) \in \Phi''_V$.

Proofs. (A) This follows from **3.1** (B) and (33), when either of reference frames is empty or either algebra is trivial. Otherwise, from (34) by (17), **3.0** and (30) $(t(U))_y = g(\chi_{V'(y)}(U)) = g(V'(y)) = g(g^{-1}(V(y))) = V(y)$ for all $y \in Y \neq \emptyset$. Hence, $t(U) = V$.

(B) In the trivial case by **3.1** (A) both reference flocks are either empty or the singleton carriers, as observed in **2.0**. Hence, the conclusion follows from (30).

Assume $X, Y \neq \emptyset$. Let $c \in \Phi'_U$, namely by **2.1** (C) $\chi_c \cdot \boldsymbol{k} = \boldsymbol{i}_A$. Then, for all $a \in A$, $g(\chi_c(\boldsymbol{k}_a)) = g(a)$ and by (31) and (26) $g(a) = \xi_{g(c)}(t(\boldsymbol{k}_a)) = \xi_{g(c)}(\boldsymbol{\kappa}_{g(a)})$. Since $g : A \twoheadrightarrow B$, we take $b = g(a)$ and get $\xi_{g(c)}(\boldsymbol{\kappa}_b) = b$ for all $b \in B$, namely $g(c) \in \Phi''_V$. Clearly, we can reverse all these implications. *Q.E.D.*

3.3 Definitions. Assume that g totally induces t and preserves both reference flocks, $a \in \Phi'_U$ iff $g(a) \in \Phi''_V$. Then, given χ and ξ, g and t induce each other as in the next characterization **3.4** (B) and we will say that our $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$ is a *description of* χ *by* ξ (see (37)) or *from* U *to* V or also a *description from the former monoid onto the latter.*

Lastly, as the preceding lemma has shown that the induced relation t is a function, we will say that t is *a matrix transformation induced by* g or *the matrix transformation induced by* it *from* χ *to* ξ or also, in case a single algebra derives both χ and ξ, *the matrix transformation induced by* g *from* U *to* V.

Notice that, if $X = Y = \emptyset$ or both algebras are trivial as in **3.1** (A), then every bijection $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$ is a description by (1) and **2.0**. In such a case, $t : \{U\} \rightarrowtail\!\!\!\twoheadrightarrow \{V\}$ is the only matrix transformation.

Consider our two general Menger systems, but with the same dimension set: χ with base $U : X \to A$ and ξ with base $V : X \to B$. We say that $n : A \rightarrowtail\!\!\!\twoheadrightarrow B$ is an *(element) renaming of* χ *by* ξ or that it *renames* χ *by* ξ *elementwise*, when it is a description of χ by ξ performing its matrix transformation $t = T_n$ columnwise, namely

$$t(M) \;=\; n \cdot M \qquad \text{and} \tag{35}$$

$$n(\chi_a(M)) \;=\; \xi_{n(a)}(n \cdot M) \text{ , for all } a \in A \text{ and } M : X \to A \text{ .} \tag{36}$$

Clearly, we could easily extend this from case $Y = X$ to the case of a bijection $l : X \rightarrowtail\!\!\!\twoheadrightarrow Y$, yet hereinafter we will omit such seeming extensions. (Our choice in **0.4** of the bases as functions allows us to permute the selectors, whereas the base sets in **0.5** require such extensions.) Notice also that, in case of an automorphism $n : A \rightarrowtail\!\!\!\twoheadrightarrow A$ of an algebra deriving our Menger systems, (36) rewrites as $\chi = \xi \cdot n$ because of the property (Endomorphism) in **1.4**.

Since in (36) χ_a and $\xi_{n(a)}$ are isomorphic, an element renaming is a (simple) case of general isomorphism [2]. Contrary to the case of vector spaces, **3.6** (A) will show a description that is not a renaming. A characterization of renamings will appear in **6.1** (A).

3.4 Corollaries. *Let* $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$ *be a description of* χ *by* ξ *as above, then*

(A) *the converse base set is made of flock combiners* $V' = g^{-1} \cdot V : Y \to \Phi'_U$*;*

(B) *when* $X, Y \neq \emptyset$, $t = T_g : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ $\boldsymbol{K}$*–induces* g *as in* **1.8** (then, T_g $\boldsymbol{K}$–induces g iff g preserves both reference flocks, because of **3.2** (B));

(C) *we can compute the operations of the (algebra of the) latter Menger system by the former,*

$$\xi_b(M) = g(\chi_{g^{-1}(b)}(t^{-1}(M)))\ ,\ \textit{for all } b \in B \textit{ and } M : Y \to B, \qquad (37)$$

while we preserve the former operations as

$$g(\chi_a(L)) = \xi_{g(a)}(t(L))\ ,\ \textit{for all } a \in A \textit{ and } L : X \to A\ ; \qquad (38)$$

(D) *descriptions define an equivalence relation among Menger systems, namely*

(Symmetry) $g^{-1}\colon B \mapsto\!\!\!\twoheadrightarrow A$ *is a description of* ξ *by* χ, *while* t^{-1} *is its matrix transformation,*

(Transitivity) *if* $h : B \mapsto\!\!\!\twoheadrightarrow C$ *is a description of* ξ *by another Menger system* γ *on* C, *then* $h \cdot g : A \mapsto\!\!\!\twoheadrightarrow C$ *is of* χ *by* γ *with the composition of their matrix transformations;*

(E) *the set of descriptions between Menger systems derived from the same algebra* α *on* A *forms a (sub)group under the functional composition on* A^A.

Proofs. (A) It is trivial for $X, Y = \emptyset$, otherwise the preservation of the reference flocks in **3.3** implies it. In fact, $g(V'(y)) = V(y) \in \Phi''_V$ for each $y \in Y$ by **2.1** (A). Hence, since $c \in \Phi'_U$ iff $g(c) \in \Phi''_V$, $V'(y) \in \Phi'_U$ for all $y \in Y$.

(B) By **1.7** (A) we can show (26). By (A) we can take any $c = V'(y) \in \Phi'_U$ for $y \in Y \neq \emptyset$ in (27) and by (34) and (9) get $((t \cdot \boldsymbol{k})(a))_y = (t(\boldsymbol{k}_a))_y = g(\chi_{V'(y)}(\boldsymbol{k}_a)) = g(a) = \boldsymbol{\kappa}_{g(a)}(y)$ for all $a \in A$ and $y \in Y$, namely $t \cdot \boldsymbol{k} = \boldsymbol{\kappa} \cdot g$.

(C) Take any $a \in A$ such that $g(a) = b$ and get (37) from (31) by **3.1** (B) and (30). To get (38), merely use **3.1** (B) on (31).

(D) (Symmetry) Total induction is symmetric as already observed in the proof of **3.1** (B) and the same holds for the preservation of the reference flocks by definition **3.3**.

(Transitivity) By **3.1** (B) both g and h induce bijections, $t = T_g : A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ and say $t' = T'_h : B^Y \mapsto\!\!\!\twoheadrightarrow C^Z$. This implies $t' \cdot t : A^X \mapsto\!\!\!\twoheadrightarrow C^Z$. Hence, to get the transitivity of total induction, we only have to prove that $h(g(\chi_a(L))) = \gamma_{h(g(a))}(t'(t(L)))$, for all $a \in A$ and $L : X \to A$. This easily follows from (38), used twice: $h(g(\chi_a(L))) = h(\xi_{g(a)}(t(L))) = \gamma_{h(g(a))}(t'(t(L)))$. Lastly, the transitivity of the preservation of reference flocks is trivial.

(E) The closure under composition was just proved in (D) (Transitivity), the composition inverse in (D) (Symmetry). As the composition was the functional one, we get the required group with unit $\boldsymbol{i}_A$. This unit is the renaming description that corresponds to $\xi = \chi$ or to $V = U$ with $t = \boldsymbol{i}_{A^X}$, yet **3.6** will show that, given $g = \boldsymbol{i}_A$, sometimes also other ξ's, V and t's can do. *Q.E.D.*

3.5 Example. (A) Given two Menger systems, by **1.4** (Menger to algebra) one might consider the isomorphisms between the algebras deriving them. In **3.3**

we did not require that $g : A \twoheadrightarrow B$ be such an isomorphism nor later we proved it was. To check that this requirement is not granted consider the classical example for semi-linear transformations [21].

Let $\chi = \xi$ be the Menger system for the complex vector space on the complex field with the usual frame of selectors (versors) $U = V : 3 \to A$, namely $\chi_a(L)$ is the usual product of vector a times matrix $L : 3 \to A$. When we define $g : A \twoheadrightarrow A$ as the componentwise complex conjugation, we have a bijection that is not an automorphism of the space (nor of χ), such that

$$g \cdot U = U \ . \tag{39}$$

Then, (34) by (19) defines $t : A^3 \to A^3$ as $(t(L))_y = g(\chi_{U(y)}(L)) = g(L(y))$ for $y = 0, 1, 2$, namely $t(L) = g \cdot L$ for all matrices $L : 3 \to A$. Because of our usual χ and ξ and of this $t : A^3 \twoheadrightarrow A^3$, any L and $M = t(L)$ easily satisfy (31) for all vectors a.

This implies that the relation induced by g in (31) contains our t, that it is totally induced and, by **3.1** (B), that it is t. Also, from **2.0** we easily see that g preserves the reference flock. Therefore, g is a description, nay a renaming as in (36), but not an isomorphism.

(B) Notice that, while the choice of the reference frames determines a single automorphism $g : A \twoheadrightarrow A$, it does not for "self-descriptions" $g : A \twoheadrightarrow A$, in spite of the dependence found in **3.1** (C). In fact, the self-description g of (A) extends the identity on the reference vectors as in (39), yet by **3.4** (E) also $g = \boldsymbol{i}_A$ does and clearly its matrix transformation is $t = \boldsymbol{i}_{A^3}$. In each case both the description and its matrix transformation differ with respect to the other case.

Here, both descriptions are renamings that still *determine the latter reference frame* by (35) and its matrix transformation by **3.1** (C). To identify a "transformation", we do not need neither matrix transformations nor reference frames and, since the latter only determines isomorphisms, descriptions alone can replace them. Yet, to define such a description we still need some condition not involving matrix transformations.

6.5 (B) will show that in a vector space our renamings are the semi-linear transformations. Hence, such a condition there is an equation that involves vector-space scalars. The corresponding dilatations do not depend on any its reference frame, as formalized in **2.8**. This will explain why in vector spaces abstract representation-free theories work.

6.4 will show that we can define our universal descriptions too by scalars through a condition formally identical to the one of semi-linear transformations. Yet, the next example will also also show that universal scalars are representation dependent. Then, the same condition, used to get rid of reference frames in vector spaces, will prove that they become mandatory in general.

3.6 Example. (A) We show that outside vector spaces there are descriptions that are not renamings. We exhibit a description between two Menger systems of a different dimension that are derived from the same algebra. We first show the existence of such an algebra and we introduce it through some algebraic conventions that later we will replace by set-theoretical ones.

Let us consider a possible algebra on a carrier A with five operations:

$$\boldsymbol{f}_0, \boldsymbol{f}_1 : (A \times A) \times A \to A \quad \text{and} \quad \boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2 : A \times A \to A \ .$$

that satisfy nine equations: for all $x : 2 \to A$, $y : 3 \to A$ and $z \in A$

$$\begin{cases} \boldsymbol{f}_0(\boldsymbol{g}_0(x_0,x_1), \boldsymbol{g}_1(x_0,x_1), \boldsymbol{g}_2(x_0,x_1)) = & x_0 \ , \\ \boldsymbol{f}_1(\boldsymbol{g}_0(x_0,x_1), \boldsymbol{g}_1(x_0,x_1), \boldsymbol{g}_2(x_0,x_1)) = & x_1 \ , \end{cases} \tag{40}$$

$$\begin{cases} \boldsymbol{g}_0(\boldsymbol{f}_0(y_0,y_1,y_2), \boldsymbol{f}_1(y_0,y_1,y_2)) = & y_0 \ , \\ \boldsymbol{g}_1(\boldsymbol{f}_0(y_0,y_1,y_2), \boldsymbol{f}_1(y_0,y_1,y_2)) = & y_1 \ , \\ \boldsymbol{g}_2(\boldsymbol{f}_0(y_0,y_1,y_2), \boldsymbol{f}_1(y_0,y_1,y_2)) = & y_2 \ , \end{cases} \tag{41}$$

$$\begin{cases} \boldsymbol{f}_0(z,z,z) & = \boldsymbol{f}_1(z,z,z) \quad \text{and} \\ \boldsymbol{g}_0(z,z) = & \boldsymbol{g}_1(z,z) = \boldsymbol{g}_2(z,z) = z \ . \end{cases} \tag{42}$$

The natural correspondences $(A \times A) \times A \simeq A^3$ and $A \times A \simeq A^2$ allow us to replace our $\boldsymbol{f} : 2 \to A^{(A \times A) \times A}$ and $\boldsymbol{g} : 3 \to A^{A \times A}$ by the functions

$$f' : 2 \quad \to \quad A^{A^3} \quad \text{and} \tag{43}$$

$$g' : 3 \quad \to \quad A^{A^2} \ , \tag{44}$$

such that $f'_j(y) = \boldsymbol{f}_j(y_0, y_1, y_2)$ and $g'_i(x) = \boldsymbol{g}_i(x_0, x_1)$ for all $j \in 2$, $i \in 3$, $x : 2 \to A$ and $y : 3 \to A$. If $\mathbf{C}_{f'}$ and $\mathbf{C}_{g'}$ denote the functions $\mathbf{C}_{f'} : A^3 \to A^2$ and $\mathbf{C}_{g'} : A^2 \to A^3$ such that $(\mathbf{C}_{f'}(y))_j = {f'}_j(y)$ and $(\mathbf{C}_{g'}(x))_i = g'_i(x)$ for all such j, i, x and y, then we can rewrite (40) and (41) respectively as

$$\mathbf{C}_{f'} \cdot \mathbf{C}_{g'} \quad = \quad \boldsymbol{i}_{A^2} \quad \text{and} \tag{45}$$

$$\mathbf{C}_{g'} \cdot \mathbf{C}_{f'} \quad = \quad \boldsymbol{i}_{A^3} \ . \tag{46}$$

Therefore, we got two functions $f'' = \mathbf{C}_{f'}$ and $g'' = \mathbf{C}_{g'}$ that are the one the inverse of the other. Conversely, any $f'' : A^3 \mapsto\!\!\!\!\twoheadrightarrow A^2$ and $g'' : A^2 \mapsto\!\!\!\!\twoheadrightarrow A^3$, with $f'' = g''^{-1}$, define an f' and a g' as in (43) and (44), such that (45) and (46) hold. Hence, they also define an $\boldsymbol{f}$ and a $\boldsymbol{g}$ that satisfy the first five equations (40) and (41).

To check that all nine equations, (40), (41) and (42), are consistent, let us define a non trivial algebra without an empty carrier, satisfying them, by defining such f'' and g'' in a way compatible with (42). To do it, we take A to be the set of natural numbers, as usual.

Let C and D denote the subsets of constants in A^2 and A^3, $C = \{x : 2 \to A \mid x_0 = x_1\}$ and $D = \{y : 3 \to A \mid y_0 = y_1 = y_2\}$. Namely, when we set $X = 2$ and $Y = 3$ in (2) and (9), $C = \{\boldsymbol{k}_z \mid z \in A\}$ and $D = \{\boldsymbol{\kappa}_z \mid z \in A\}$. Then, we set $\bar{C} = A^2 \setminus C$ and $\bar{D} = A^3 \setminus D$ to get two bi-partitions $\{C, \bar{C}\}$ and $\{D, \bar{D}\}$ such that $A^2 = C \cup \bar{C}$ and $A^3 = D \cup \bar{D}$.

Clearly, we got two pairs of denumerable sets that allow us to take the bijection $d : C \rightarrowtail\!\!\!\!\twoheadrightarrow D$, such that $d(\boldsymbol{k}_z) = \boldsymbol{\kappa}_z$ for all $z \in A$, and to choose some bijection $e : \bar{C} \rightarrowtail\!\!\!\!\twoheadrightarrow \bar{D}$. Then, if we set $f'' = d^{-1} \cup e^{-1}$ and $g'' = d \cup e$, we get the required bijections, $f'' : A^3 \rightarrowtail\!\!\!\!\twoheadrightarrow A^2$ and $g'' : A^2 \rightarrowtail\!\!\!\!\twoheadrightarrow A^3$.

In fact, since $f'' = g''^{-1}$, we get the first five equations. Moreover, by (2) $\boldsymbol{f}_0(z,z,z) = f'_0(\boldsymbol{\kappa}_z) = (f''(\boldsymbol{\kappa}_z))_0 = (d^{-1}(\boldsymbol{\kappa}_z))_0 = \boldsymbol{k}_z(0) = z = \boldsymbol{k}_z(1) = (d^{-1}(\boldsymbol{\kappa}_z))_1 = (f''(\boldsymbol{\kappa}_z))_1 = f'_1(\boldsymbol{\kappa}_z) = \boldsymbol{f}_1(z,z,z)$ for all $z \in A$, while the remaining equations follows from (9) in the same way: e.g. $\boldsymbol{g}_0(z,z) = g'_0(\boldsymbol{k}_z) = (g''(\boldsymbol{k}_z))_0 = (d(\boldsymbol{k}_z))_0 = \boldsymbol{\kappa}_z(0) = z = \boldsymbol{\kappa}_z(1) = (d(\boldsymbol{k}_z))_1 = (g''(\boldsymbol{k}_z))_1 = g'_1(\boldsymbol{k}_z) = \boldsymbol{g}_1(z,z)$ for all $z \in A$,

Since there is such an algebra satisfying all nine equations, there also is a free one, satisfying them with a doubleton base set. Since any such a free algebra satisfies the first five equations (40) and (41), by the theorem of A. Goetz and C. Ryll-Nardzewski [4] as in **5** §31 of [6] it has bases with two and more reference elements.

Now, let A and $U : 2 \to A$ respectively denote the carrier of such an algebra and one of these doubleton bases. Then, the former of (13) with $X = 2$ represents the set of endomorphisms $\mathcal{E}$ of this based algebra, as well as of the algebra of the Menger system with respect to any base as in (24). This will serve to prove that *the identity is a description between the Menger systems derived with respect to U and to another base with three reference elements.*

Proof. Notice that the natural correspondence $A \times A \simeq A^2$ rewrites the endomorphic condition $h(\boldsymbol{g}_i(x_0, x_1)) = \boldsymbol{g}_i(h(x_0), h(x_1))$ for $h \in \mathcal{E}$ and $x : 2 \to A$ as $h(g'_i(x)) = g'_i(h \cdot x)$. Set $V = g''(U)$. Then $V : 3 \to A$ and $\boldsymbol{r}''_V : \mathcal{E} \to A^3$, where we claim that $\boldsymbol{r}''_V$ is a bijection as in the latter of (13) with $B = A$, $\mathcal{F} = \mathcal{E}$ and $Y = 3$. In fact, since $(\boldsymbol{r}''_V(h))_i = (h \cdot g''(U))_i = h((g''(U))_i) = h(g'_i(U)) = g'_i(h \cdot U) = g'_i(\boldsymbol{r}'_U(h)) = (g''(\boldsymbol{r}'_U(h)))_i$ for all $h \in \mathcal{E}$ and $i \in 2$, it is the composition of two bijections:

$$\boldsymbol{r}''_V = g'' \cdot \boldsymbol{r}'_U \tag{47}$$

This composition also allows the Menger system $\chi : A \to A^{A^2}$ with respect to U to redefine the one $\xi : A \to A^{A^3}$ with respect to Vas

$$\xi_a = \chi_a \cdot f'' \ , \qquad \text{for all } a \in A \ . \tag{48}$$

In fact, from (47) by (45) and (46) we get $\boldsymbol{r}'_U = g''^{-1} \cdot \boldsymbol{r}''_V = f'' \cdot \boldsymbol{r}''_V$. Then, by (15), (24) and **0.4**, for all $M = \boldsymbol{r}''_V(e) : 3 \to A$ and $a \in A$, $\xi_a(M) = e(a) = \chi_a(\boldsymbol{r}'_U(e)) = \chi_a(f''(\boldsymbol{r}''_V(e))) = \chi_a(f''(M))$.

By (45) and (46) we can also rewrite (48) as $\chi_a = \xi_a \cdot g''$. Therefore, $g = \boldsymbol{i}_A : A \Vdash\!\!\twoheadrightarrow A$ totally induces a relation containing g''. Since $g'' : A^2 \Vdash\!\!\twoheadrightarrow A^3$, by **3.1** (B) this relation is g'' itself, $T_g = g''$. Moreover, g'' $\boldsymbol{K}$–induces g as in (26), because our free algebra satisfies (42): $((g'' \cdot \boldsymbol{k})(z))_i = (g''(\boldsymbol{k}_z))_i = g'_i(\boldsymbol{k}_z) = \boldsymbol{g}_i(z,z) = z = \boldsymbol{\kappa}_z(i) = ((\boldsymbol{\kappa} \cdot g)(z))_i$ for all $z \in A$ and $i \in 3$. Hence, by **3.2** (B) the identity $g = \boldsymbol{i}_A$ is a description of χ by ξ and g'' is its matrix transformation. *Q.E.D.*

(B) Such a g also is an automorphism as well as a general one as in **2** of [2]. Yet, in both cases the renaming condition (35) fails to provide U with the new base $V = g''(U)$ we found. It gets U again, by the matrix transformation $\boldsymbol{i}_{A^2} : A^2 \Vdash\!\!\twoheadrightarrow A^2$. Descriptions alone can miss some "transformation". This also shows that the dependence in **3.1** (C) of T on V is effective.

In **3.5** (B) were the bases to miss some "transformation". To get all of them in general vector spaces, linear transformations had to generalize into semi-linear ones. Therefore, also now we need a further generalization: our descriptions with their matrix transformations might be one. (To say it is the one, we will need to compare it with other such generalizations.)

Both semi-linear transformations and general isomorphisms are representation-free notions. If one could disregard the reference systems even in the universal case, then one would say that the present algebra, not just its former representation, has 2 as a dimension, whereas it does not. In fact, the description we found changes 2 into 3, in spite of the preservations in **3.4** (C). It does, *because it is from U to V.*

The coordinate-free (or in general representation-free) approach worked in vector spaces, also because of a yet unproved (see **6.5** (A)) property: all transformations can change reference frames only by the renaming condition. Anyway, we found that in the universal case this fails. Universal generality conflicts against this approach.

(We still accept the proposal of K. Głazek to call "general" the isomorphism that E. Marczewski called "weak". In fact, as an "isomorphism", an abstract algebraic notion, it remains the most general found so far.)

3.7 Transformation groupoids. Given a free algebra, **3.4** (E) provides its "self-descriptions" with a group, alike to the one of linear or semi-linear transformations for a vector space. Yet, contrary to the vector space case, its elements can miss some essential "transformation", as remarked in **3.6** (B) for the identity description and a dimension invariance. Therefore, a possible future theory of the universal transformations, introduced by this paper, should use a structure different from this group.

If one wants to keep our notion of description, *that corresponds to the vector space transformation,* one of the possible structures might be a category: the one with bases as objects and with the triples $\langle U, g, V\rangle$, where g is a description from U to V, as morphisms. Since in **0.4** a base identifies its Menger system, this structure can avoid the missing transformation problem because of **3.1** (C). Moreover, the proof of **3.4** (D) (Transitivity) still proves associativity, while the units are trivial. By **3.4** (D) (Symmetry) we get a groupoid.

Still, another category might deserve the name of *transformation groupoid:* the one where the matrix transformations replace the descriptions in the former category. By **3.4** (B) matrix transformations identify descriptions. They form a groupoid under functional composition (with the restriction of the source and target objects) for the same reasons as before. Yet, now we can see the objects as the units of the monoids of the matrices by **1.1** (Analytic). This agrees with the finding in **0.4** of [18] that the objects naturally rising from the applications are (universal) matrices rather than their algebras.

4 Analytic transformations.

4.0 Definition. While section **3** concerned two Menger systems, this section concerns the two analytic monoids of **1.2**. Again, we introduce some preliminary notions and results to define their transformations in **4.5**.

Given a bijection $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$, consider the relation $g \subseteq A \times B$ defined for all $a \in A$ and $b \in B$ by $\langle a, b\rangle \in g$ iff

$$t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_b \quad \text{for all } L : X \to A \ , \tag{49}$$

namely g relates a and b when bijection t isomorphically relates the two unary operations on A^X and B^Y of "right product by the corresponding constant": $\chi'_a : A^X \to A^X$ and $\xi'_b : B^Y \to B^Y$ such that $\chi'_a(L) = L \circ \boldsymbol{k}_a$ and $\xi'_b(M) = M \diamond \boldsymbol{\kappa}_b$ for all $L : X \to A$ and $M : Y \to B$. Such a right product also occurred in the axioms of definition **1.0** of analytic monoids.

If g relates any $a \in A$ with some $b \in B$ and, conversely, any b with some a, then we say that t *totally induces* g *from* A *to* B or that g is the *relation totally induced by* t *from* A *to* B. Since our functions $\chi' : A \to A^{X^{A^X}}$ and $\xi' : B \to B^{Y^{B^Y}}$ define *unary* algebras on A^X and B^Y respectively, total induction requires that they are *generally* isomorphic algebras as in **2** of [2].

When $X, Y \neq \emptyset$, t determines A and B. Then, we merely say that g is the *relation totally induced by* t and we write $g = G_t$, where we denote the function relating the t's to the g's by $G \subseteq (B^Y)^{A^X} \times \mathsf{P}(A \times B)$.

The requirement that t is such a bijection again implies that singleton carriers coexist as in (33). Then, both analytic monoids are trivial and total induction by (49) defines $g = A \times B$, where by **1.0** A and B are only required to be either both empty or both nonempty: e.g. it prevents that $A^X = \emptyset^\emptyset$ and $B^Y = 1^2$, as **3.1** (A) did.

This agrees with the behavior of trivial analytic monoids in **1.0**, whereas it disagrees with the premise $g : A \mapsto\!\!\!\twoheadrightarrow B$ of **3.1** (A). Yet, the corresponding conclusions still hold. Anyway, if one of the monoid carriers is not singleton, then both $X, Y \neq \emptyset$.

4.1 Lemmata. *If* $t : A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ *totally induces a relation from* A *to* B, $g \subseteq A \times B$, *then*

(A) *t retypes $\boldsymbol{K}$ as in* **1.6** *and,*

(B) *when $X, Y \neq \emptyset$, the induced relation is a bijection, $g = G_t : A \mapsto\!\!\!\twoheadrightarrow B$.*

Proofs. (A) When the carriers are singleton, it follows from $t = t \cdot \boldsymbol{i}_C$ as in **1.6**. Otherwise, the dimensions are not trivial and by **1.0** $A, B \neq \emptyset$. Since for every $a \in A$ there is some $b \in B$ that satisfies (49), from (4), (49) and (8) we get for all $a \in A$ that $t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = t(U) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_{\xi_b(t(U))} = \boldsymbol{\kappa}_{b'}$ for some $b' = \xi_b(t(U))$. Conversely, for all $b \in B$ from (11), (49) and (7) we get $\boldsymbol{\kappa}_b = V \diamond \boldsymbol{\kappa}_b = t(t^{-1}(V)) \diamond \boldsymbol{\kappa}_b = t(t^{-1}(V) \circ \boldsymbol{k}_a) = t(\boldsymbol{k}_{\chi_a(t^{-1}(V))}) = t(\boldsymbol{k}_{a'})$ for some $a' = \chi_a(t^{-1}(V))$ with $\langle a, b \rangle \in g$.

(B) Let us show that the induced relation is a function, $g = G_t : A \to B$. Let $\langle a, b' \rangle, \langle a, b'' \rangle \in g$. In (49) take $L = t^{-1}(V)$. Then, $\boldsymbol{\kappa}_{b'} = V \diamond \boldsymbol{\kappa}_{b'} = t(L) \diamond \boldsymbol{\kappa}_{b'} = t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_{b''} = V \diamond \boldsymbol{\kappa}_{b''} = \boldsymbol{\kappa}_{b''}$, because of (11). Hence, (9) gets $b' = b''$.

Since $g : A \twoheadrightarrow B$ comes from the total induction assumption, now we only have to show that g^{-1} too is a function, $g^{-1} : B \to A$. This, easily follows from (4) and (2) by the converse of the preceding reasoning. In fact, $\langle a', b \rangle, \langle a'', b \rangle \in g$ implies $t(\boldsymbol{k}_{a'}) = t(U \circ \boldsymbol{k}_{a'}) = t(U) \diamond \boldsymbol{\kappa}_b = t(U \circ \boldsymbol{k}_{a''}) = t(\boldsymbol{k}_{a''})$ by (49) with $L = U$. Since t is one to one, this implies $\boldsymbol{k}_{a'} = \boldsymbol{k}_{a''}$, whence $a' = a''$. *Q.E.D.*

4.2 Corollary. *If $t : A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ totally induces the bijection g of* **4.1** (B) *for $X, Y \neq \emptyset$ and preserves the unit,*

$$t(U) = V \ , \tag{50}$$

then g is the $\boldsymbol{K}$–induced bijection of **1.8***: for all $a \in A$ and every $y \in Y$, $g(a) = G_t(a) = t(\boldsymbol{k}_a)(y)$, namely $t(\boldsymbol{k}_a) = \boldsymbol{\kappa}_{g(a)}$.*

Proof. Take $L = U$ in (49). Then, for every $a \in A$ by (4), (50) and (11) get $t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = V \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b$, where by **4.1** (B) $b = g(a)$. Hence, by (9) $t(\boldsymbol{k}_a)(y) = \boldsymbol{\kappa}_{g(a)}(y) = g(a)$ for every $y \in Y$. *Q.E.D.*

4.3 Definition. Given the two analytic monoids of **1.2**, we say that $t : A^X \Vdash\!\!\twoheadrightarrow B^Y$ *preserves* $\boldsymbol{K}$*–restricted products* when

$$t(L \circ \boldsymbol{k}_a) = t(L) \diamond t(\boldsymbol{k}_a) \text{ for all } L : X \to A \text{ and } a \in A \ . \tag{51}$$

4.4 Lemma. *If $t : A^X \Vdash\!\!\twoheadrightarrow B^Y$ totally induces a relation from A to B and preserves the unit, then it preserves $\boldsymbol{K}$–restricted products.*

Proof. Trivial for singleton carriers. Otherwise $X, Y \neq \emptyset$. Then, start from (49) and use **4.1** (B) and **4.2**: for all $L : X \to A$ and $a \in A$, $t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_b = t(L) \diamond \boldsymbol{\kappa}_{g(a)} = t(L) \diamond t(\boldsymbol{k}_a)$. *Q.E.D.*

4.5 Definitions. Consider the two analytic monoids of **1.2** and a bijection between their carriers, $t : A^X \Vdash\!\!\twoheadrightarrow B^Y$. The conditions of total induction and unit preservation are enough to get the preservation of other features between such analytic monoids, as we have just shown and we will also find in **4.6**. Hence,we will say that t is an *analytic transformation from* the former monoid *to* the latter when it totally induces g from A to B as in **4.0** and preserves the unit as in (50). However, even the two preservation properties, we have shown in the preceding lemmata, are enough and will allow us to use the following characterization **4.6** (A).

When $X, Y \neq \emptyset$, the two analytic monoids identify the two Menger systems of **1.2**, while t can be the subject of the depiction property (26). Then, we will say that g, the bijection $\boldsymbol{K}$-induced by t as in **4.2**, is the *analytic description of χ by ξ* or *from* the former monoid *onto the latter.*

When $Y = \emptyset$, both the intensional expression of G in **4.2** and the one of T in (34) fail to express g and t respectively, though both **4.2** and **3.1** (C) are true. Yet, contrary to matrix transformations, analytic descriptions are not defined, because of the set-theoretical reason in the note of **1.6**.

4.6 Theorems.

(A) *When both dimensions are not trivial, $t : A^X \Vdash\!\!\twoheadrightarrow B^Y$ is an analytic transformation iff it retypes $\boldsymbol{K}$ as in* **1.6** *and preserves $\boldsymbol{K}$-restricted products as in (51).*

(B) *An analytic transformation t is a monoid isomorphism, namely it preserves the units, $t(U) = V$, and the matrix product, $t(M \circ L) = t(M) \diamond t(L)$ for all $L, M : X \to A$.*

Proofs. (A) (Only if) $\boldsymbol{K}$–retyping comes from **4.1** (A), while the other preservation comes from **4.4**. (Hence, this holds even for $X = \emptyset$ or $Y = \emptyset$.)

(If) From (51) and **1.7** (A) we get $t(L \circ \boldsymbol{k}_a) = t(L) \diamond t(\boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_{g(a)}$ for some $g : A \Vdash\!\!\twoheadrightarrow B$ and all $L : X \to A$ and $a \in A$. Hence, t totally induces some relation, which by (49) and **4.1** (B) is this g. For $L = U$ this also implies that, for all $b = g(a) \in B$, $\boldsymbol{\kappa}_b = t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = t(U) \diamond \boldsymbol{\kappa}_b$ by (4). Then, by (8) and (9) $\xi_b(t(U)) = b$ for all $b \in B$, which by **1.4** and (13) states that $t(U) = \boldsymbol{r}''_V(\boldsymbol{i}_B)$ as in **0.4**, namely $t(U) = V$.

(Notice that, when some dimension is trivial, say $X = \emptyset$, the preservation of the unit still comes from (33) as observed in **4.0**, whereas total induction fails for $A = \emptyset$ and $B \neq \emptyset$.)

(B) t preserves the units by definition. It also trivially preserves the matrix product in the singleton carrier case. Hence, we can assume $X, Y \neq \emptyset$ and, for all $L, M : X \to A$ and $y \in Y$, in order to prove $(t(M \circ L))_y = (t(M) \diamond t(L))_y$, we prove $\boldsymbol{\kappa}_{(t(M \circ L))_y} = \boldsymbol{\kappa}_{(t(M) \diamond t(L))_y}$ because of (25).

In fact, we use (10), **4.1** (A), **4.2**, **4.4**, (5), **4.4**, **4.2**, (10), **4.1** (A), (51), (10), (12) and (10) to get, $\boldsymbol{\kappa}_{(t(M \circ L))_y} = t(M \circ L) \diamond \boldsymbol{\kappa}_{V(y)} = t(M \circ L) \diamond t(\boldsymbol{k}_{V'(y)}) = t((M \circ L) \circ \boldsymbol{k}_{V'(y)}) = t(M \circ (L \circ \boldsymbol{k}_{V'(y)})) = t(M \circ t^{-1}(t(L \circ \boldsymbol{k}_{V'(y)}))) = t(M \circ t^{-1}(t(L) \diamond \boldsymbol{\kappa}_{V(y)}))) = t(M \circ t^{-1}(\boldsymbol{\kappa}_{(t(L))_y})) = t(M) \diamond t(t^{-1}(\boldsymbol{\kappa}_{(t(L))_y})) = t(M) \diamond \boldsymbol{\kappa}_{(t(L))_y} = t(M) \diamond (t(L) \diamond \boldsymbol{\kappa}_{V(y)}) = (t(M) \diamond t(L)) \diamond \boldsymbol{\kappa}_{V(y)} = \boldsymbol{\kappa}_{(t(M) \diamond t(L))_y}$. *Q.E.D.*

4.7 Corollaries.

(A) *An analytic transformation t preserves the scalars in both ways: for all $S : X \to A$*

$$t(S) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b \diamond t(S) \text{ for all } b \in B \text{ iff } S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S \text{ for all } a \in A \ , \quad (52)$$

according to characterization **2.6** (A). Hence, $t \cdot \boldsymbol{i}_F : F \Vdash\!\!\twoheadrightarrow G$ is an isomorphism between scalar monoids by **2.7** (B) and **4.6** (B).

(B) *When $X, Y \neq \emptyset$, t and its analytic description g preserve the derived Menger systems as in (38).*

Proofs. (A) In case of singleton carriers, the unit scalar is the only matrix and the statement is obvious. Otherwise, we only have to prove (52) for $X, Y \neq \emptyset$. By **4.6** (B) $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$ iff $t(S) \diamond t(\boldsymbol{k}_a) = t(\boldsymbol{k}_a) \diamond t(S)$ for all $a \in A$ and by **4.2** iff $t(S) \diamond \boldsymbol{\kappa}_{g(a)} = \boldsymbol{\kappa}_{g(a)} \diamond t(S)$ for all $a \in A$, which by **4.1** (B) is equivalent to $t(S) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b \diamond t(S)$ for all $b \in B$.

(B) Take some $y \in Y$. By **4.2**, (7), **4.6** (A), **4.2**, (8) and (9) we get $g(\chi_a(L)) = t(\boldsymbol{k}_{\chi_a(L)})(y) = t(L \circ \boldsymbol{k}_a)(y) = (t(L) \diamond t(\boldsymbol{k}_a))(y) = (t(L) \diamond \boldsymbol{\kappa}_{g(a)})(y) = \boldsymbol{\kappa}_{\xi_{g(a)}(t(L))}(y) = \xi_{g(a)}(t(L))$ for all $a \in A$ and $L : X \to A$. *Q.E.D.*

4.8 Lemmata. Let g be the analytic description for t as in **4.5** and consider the two derived Menger systems, then

(A) *g is a centralizer bijection: for all $e : A \to A$ and $f : B \to B$ such that $g \cdot e = f \cdot g$,*

$$e \in \mathcal{E} \ \text{ iff } \ f \in \mathcal{F} \ ; \quad (53)$$

(B) *$c \in A$ is a dilatation indicator (in the former Menger system) iff $g(c) \in B$ is (in the latter).*

Proofs. (A) By (14) and (15) we can prove that, when

$$g(e(a)) \;=\; f(g(a)) \text{ for all } a \in A\,, \tag{54}$$

for each $L : X \to A$, such that

$$e(a) \;=\; \chi_a(L) \text{ for all } a \in A\,, \tag{55}$$

there is an $M : Y \to B$, such that

$$\xi_b(M) \;=\; f(b) \text{ for all } b \in B\,, \tag{56}$$

and — conversely — for each such an M there is such an L. Since $g : A \rightarrowtail B$, we can replace (55) by $g(e(a)) = g(\chi_a(L))$ for all $a \in A$. Since $g : A \twoheadrightarrow B$, we can replace (56) by $\xi_{g(a)}(M) = f(g(a))$ for all $a \in A$.

Therefore, because of (54), we only have to prove that for each L there is an M and for each M there is an L such that $g(\chi_a(L)) = \xi_{g(a)}(M)$ for all $a \in A$. This is what our relation $t : A^X \rightarrowtail\!\!\!\!\twoheadrightarrow B^Y$ does by **4.7** (B), when we set $M = t(L)$ in (38).

(B) By **2.3** any c is a dilatation indicator iff there is $L : X \to A$ such that $\chi_c(\boldsymbol{k}_a) = \chi_a(L)$ for all $a \in A$. As $g : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$, this occurs iff $g(\chi_c(\boldsymbol{k}_a)) = g(\chi_a(L))$ for all $a \in A$, namely by **4.7** (B) and **4.2** iff $\xi_{g(c)}(\boldsymbol{\kappa}_{g(a)}) = \xi_{g(a)}(t(L))$ for all $a \in A$. Since both $t : A^X \rightarrowtail\!\!\!\!\twoheadrightarrow B^Y$ and $g : A \twoheadrightarrow B$, we can set $M = t(L)$ and $b = g(a)$ to rewrite it as $\xi_{g(c)}(\boldsymbol{\kappa}_b) = \xi_b(M)$ for all $b \in B$.

Hence, c is a dilatation indicator iff there is $M : Y \to B$ such that the last condition holds. By **2.3** this occurs iff $d = g(c) \in B$ is a dilatation indicator. *Q.E.D.*

5 Geometric descriptions and transformations.

5.0 Definition. Consider two representations for based algebras as in (13) that derive our Menger systems χ and ξ by **0.4**. Given any $g : A \to B$, let $\widetilde{g}$ denote the function that indexes relations by endomorphisms, $\widetilde{g} : \mathcal{E} \to \mathsf{P}(B \times B)$, defined for all $e \in \mathcal{E} \subseteq A \times A$ by

$$\widetilde{g}_e = \{\langle g(a'), g(a'')\rangle \mid \langle a', a''\rangle \in e\}\ . \tag{57}$$

Namely, $\widetilde{g}_e$ is the "image" of e under g. We will call it the *g–image of e.*

5.1 Lemmata. *If $g : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$, then*

(A) $\widetilde{g} : \mathcal{E} \rightarrowtail B^B$,

(B) *every g–image of an endomorphism is its "g–transformed", i.e. $\widetilde{g}_e = g \cdot e \cdot g^{-1}$, for all $e \in \mathcal{E}$, that implies that*

(C) $\widetilde{g}_e(g(a)) = g(e(a))$ *for all* $a \in A$,

(D) *g–images preserve compositions,* $\widetilde{g}_{e''\cdot e'} = \widetilde{g}_{e''} \cdot \widetilde{g}_{e'}$ *, for all* $e', e'' \in \mathcal{E}$ *, and that*

(E) $\widetilde{g}_{\boldsymbol{i}_A} = \boldsymbol{i}_B$.

Proofs. (B and A) As $e : A \to A$, if we set $a' = a$ in (57), we can rewrite it as

$$\widetilde{g}_e = \{\langle g(a), g(e(a))\rangle \mid a \in A\} \ .$$

It follows that $\widetilde{g}_e \cdot g = g \cdot e$. Since $g^{-1} : B \mapsto\!\!\!\twoheadrightarrow A$, we get $g \cdot e \cdot g^{-1} = \widetilde{g}_e \cdot (g \cdot g^{-1}) = \widetilde{g}_e$. Hence, $\widetilde{g}_e : B \to B$ for all $e \in \mathcal{E}$, because compositions of functions are functions. This also shows that $\widetilde{g}$ has to be one to one, because $\widetilde{g}_{e'} = \widetilde{g}_{e''}$ by the bijectivities of g and g^{-1} implies $e' = e''$.

(C) It follows from $\widetilde{g}_e : B \to B$ and from $\widetilde{g}_e \cdot g = g \cdot e$ as above.

(D) Trivial computations: $\widetilde{g}_{e''\cdot e'} = g \cdot e'' \cdot e' \cdot g^{-1} = g \cdot e'' \cdot g^{-1} \cdot g \cdot e' \cdot g^{-1} = \widetilde{g}_{e''} \cdot \widetilde{g}_{e'}$ for all $e', e'' \in \mathcal{E}$.

(E) Immediate from $g : A \twoheadrightarrow B$ and (57). *Q.E.D.*

5.2 Definitions. We say that a bijection $g : A \mapsto\!\!\!\twoheadrightarrow B$ *fully preserves dilatations* when the g–images preserve all dilatations in both ways, $\widetilde{g}_e$ is a dilatation of ξ iff e is of χ, while g preserves the "amount" of the dilatation involved by preserving the indicators in both ways, viz. $\chi_c \cdot \boldsymbol{k} = e \in \mathcal{E}$ iff $\widetilde{g}_e = \xi_{g(c)} \cdot \boldsymbol{\kappa} \in \mathcal{F}$.

We say that $g : A \mapsto\!\!\!\twoheadrightarrow B$ is a *geometric description of* χ *by* ξ or *from the representation of* $\mathcal{E}$ *by* U *to the one of* $\mathcal{F}$ *by* V, when g fully preserves dilatations, $\chi_c \cdot \boldsymbol{k} \in \mathcal{E}$ iff $\xi_{g(c)} \cdot \boldsymbol{\kappa} = \widetilde{g}_{\chi_c \cdot \boldsymbol{k}} \in \mathcal{F}$, while the g–images preserve all endomorphisms in both ways, $\widetilde{g} : \mathcal{E} \twoheadrightarrow \mathcal{F}$. The adjective "geometric" refers to the next property **5.4** and to its corollaries **6.1** (C) and (D) (used in **6.2** to show that in vector spaces descriptions induce projectivities).

In such a case $\widetilde{g} : \mathcal{E} \mapsto\!\!\!\twoheadrightarrow \mathcal{F}$ by **5.1** (A). We will call it a *geometric transformation from* the representation of $\mathcal{E}$ by U *to* the one of $\mathcal{F}$ by V, as in (13). As mentioned in **0.5**, it is not necessary to assume two algebras. We can well start only from certain composition submonoids on some $\mathcal{E} \subseteq A^A$ and on some $\mathcal{F} \subseteq B^B$ and get the algebras or Menger systems from (0).

5.3 Corollary. *Let* $g : A \mapsto\!\!\!\twoheadrightarrow B$ *be a geometric description as above. Then, the two trivial dimensions must coexist,* $X = \emptyset$ *iff* $Y = \emptyset$*, or both* A *and* B *are singleton. Hence, (33) holds and, when the sets of matrices are singleton, every bijection from* A *onto* B *is such a* g.

Proof. The coexistence of singletons comes from $g : A \mapsto\!\!\!\twoheadrightarrow B$. Then, consider dimension triviality without singletons. Since by (13) $\boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U : A^X \mapsto\!\!\!\twoheadrightarrow B^Y$, trivial dimensions must coexist. In this case, any $g : A \mapsto\!\!\!\twoheadrightarrow B$ is a geometric description, because the only dilatations and endomorphisms are the two identities, both with or both without indicators as in **2.0** or **2.2** respectively. *Q.E.D.*

5.4 Theorem. *A geometric description preserves flock combiners in both ways: $c \in \Phi'_U$ iff $g(c) \in \Phi''_V$.*

Proof. Let c be a flock combiner of χ, $\chi_c \cdot \boldsymbol{k} = \boldsymbol{i}_A$. As $\boldsymbol{i}_A \in \mathcal{E}$, $\widetilde{g}_{\boldsymbol{i}_A} \in \mathcal{F}$ is a dilatation of ξ and $g(c)$ is one of its indicator, because g fully preserves dilatations. By **5.1** (E) it has to be an indicator of the identity, $\xi_{g(c)} \cdot \boldsymbol{\kappa} = \boldsymbol{i}_B$. Hence, $g(c)$ is a flock combiner of ξ. Conversely, since $g : A \Vdash\!\twoheadrightarrow B$, we can start with any such flock combiner $g(c)$ and, since $\widetilde{g} : \mathcal{E} \Vdash\!\twoheadrightarrow \mathcal{F}$, we can reverse the above passages to get that c is a flock combiner of χ. By **2.1** (C) we can also says that g preserves reference flocks in both ways. *Q.E.D.*

6 The triple characterization.

6.0 Theorem. *When the bases or units are not trivial, all three notions of description, as well as of transformation, are the same, namely g is a description iff it is analytic and iff it is geometric, while the corresponding matrix and analytic transformations t are the same and correspond to the geometric one: $t(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E}$. In the trivial case this holds for the two descriptions and for the three transformations.*

Proofs. At first, we consider $X, Y \neq \emptyset$.

(description $\Rightarrow$ analytic) Let us show that, given a description $g : A \Vdash\!\twoheadrightarrow B$, its matrix transformation t is an analytic transformation between the derived analytic monoids. We use characterization **4.6** (A). By **3.1** (B) it is a bijection $t : A^X \Vdash\!\twoheadrightarrow B^Y$. It also retypes $\boldsymbol{K}$ by **1.7** (A). In fact, by **3.1** (C), **3.4** (A), (27) and (9) $t(\boldsymbol{k}_a)(y) = g(\chi_{V'(y)}(\boldsymbol{k}_a)) = g(a) = \boldsymbol{\kappa}_{g(a)}(y)$, for all $y \in Y$ and $a \in A$, i.e. (26) holds. Moreover, this shows that t $\boldsymbol{K}$–induces our description.

Lastly, the preservation of $\boldsymbol{K}$–restricted multiplications comes from the properties (Monoid to Menger) and (Menger loop) of **1.4** by using (7), (38), (8) and (26) twice. In fact, for all $a \in A$ and $L : X \to A$, $t(L \circ \boldsymbol{k}_a) = t(\boldsymbol{k}_{\chi_a(L)}) = \boldsymbol{\kappa}_{g(\chi_a(L))} = \boldsymbol{\kappa}_{\xi_{g(a)}(t(L))} = t(L) \diamond \boldsymbol{\kappa}_{g(a)} = t(L) \diamond t(\boldsymbol{k}_a)$. As t is an analytic transformation that $\boldsymbol{K}$–induces g, a description has to be an analytic one.

(analytic $\Rightarrow$ geometric) Since the units are not trivial, by **1.2** there only is one pair of based algebras χ with base U and ξ with V, which are derived from the two analytic monoids as Menger systems. Keep g and notice that by **5.1** (B) we can rewrite the premise $g \cdot e = f \cdot g$ in lemma **4.8** (A) as $\widetilde{g}_e = f$, since $g : A \Vdash\!\twoheadrightarrow B$. Hence this lemma tells us that $\widetilde{g}$ preserves all endomorphisms in both ways.

To check the full preservation of dilatations, let us start with a dilatation $e = \chi_c \cdot \boldsymbol{k} : A \to A$ of χ, for any dilatation indicator $c \in A$. Consider $\widetilde{g}_{\chi_c \cdot \boldsymbol{k}} = f$. Since $g : A \Vdash\!\twoheadrightarrow B$, by **5.1** (B) $g \cdot \chi_c \cdot \boldsymbol{k} = f \cdot g$. Then, for all $a \in A$, $f(g(a)) = (f \cdot g)(a) = (g \cdot \chi_c \cdot \boldsymbol{k})(a) = g(\chi_c(\boldsymbol{k}_a)) = \xi_{g(c)}(t(\boldsymbol{k}_a)) = \xi_{g(c)}(\boldsymbol{\kappa}_{g(a)})$ because of **4.7**

(B) and **4.2**. As $g: A \twoheadrightarrow B$, we can rewrite it as $f(b) = \xi_{g(c)}(\boldsymbol{\kappa}_b) = (\xi_{g(c)} \cdot \boldsymbol{\kappa})(b)$ for all $b \in B$. Hence, $f = \xi_{g(c)} \cdot \boldsymbol{\kappa}$, where $g(c) = d$ has to be a dilatation indicator because of **4.8** (B), namely f is a dilatation of ξ.

Conversely, given any dilatation $f = \xi_d \cdot \boldsymbol{\kappa}$ of ξ, we can set $d = g(c)$, since $g: A \twoheadrightarrow B$. By reversing the above passages we can use **4.8** (B) again to find that c is a dilatation indicator defining the above dilatation e of χ with $\widetilde{g}_e = f$.

Lastly, let us check that the geometric transformation $\widetilde{g}: \mathcal{E} \mapsto\!\!\twoheadrightarrow \mathcal{F}$, we found, is the one corresponding to our starting analytic transformation $t: A^X \mapsto\!\!\twoheadrightarrow B^Y$. Namely, when e denotes the endomorphism of χ corresponding to a matrix $L = e \cdot U: X \to A$, i.e. by (14) $e(a) = \chi_a(L)$ for all $a \in A$, the endomorphism $\widetilde{g}_e$ of ξ has to correspond to $t(L)$, i.e. $t(L) = \widetilde{g}_e \cdot V$ or by (15) $\widetilde{g}_e(b) = \xi_b(t(L))$ for all $b \in B$. This immediately comes from **4.7** (B). In fact, by (38) $\widetilde{g}_e(b) = g(e(g^{-1}(b))) = g(\chi_{g^{-1}(b)}(L)) = \xi_{g(g^{-1}(b))}(t(L)) = \xi_b(t(L))$ for all $b \in B$.

(geometric $\Rightarrow$ description) Keep g and the derived Menger systems. By **5.4** and **2.1** (C) we only have to show that g totally induces some t and that $\widetilde{g}$ corresponds to t. We can do it first by defining a $t': A^X \twoheadrightarrow B^Y$, such that it corresponds to $\widetilde{g}$, and then by checking that $t' \subseteq t$ (which implies $t' = t$ by **3.1** (B)).

We write the former correspondence as $t'(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E}$. Since $\widetilde{g}_e \in \mathcal{F}$, this serves to define a $t': A^X \to B^Y$, because we can rewrite it as $t'(\boldsymbol{r}'_U(e)) = \widetilde{g}_e \cdot V = \boldsymbol{r}''_V(\widetilde{g}_e)$ for all $e \in \mathcal{E}$ and get $t' = \boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U$ by (13).

By (13) and **5.2** the function $t' = \boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U$ is a bijection onto B^Y, $t': A^X \mapsto\!\!\twoheadrightarrow B^Y$. Let us show that all pairs $\langle L, M \rangle \in t'$ satisfy (31). Any such a pair is in t' when there is an $e \in \mathcal{E}$, such that $L = e \cdot U$ and $M = \widetilde{g}_e \cdot V$. By (14) and (15) it satisfies (31) when such an e and $\widetilde{g}_e$ satisfy $g(e(a)) = \widetilde{g}_e(g(a))$ for all $a \in A$. This was proved in **5.1** (C).

(Trivial case) To check that a geometric description is a description between the derived Menger systems and conversely, note that the derivation of a Menger system from a based algebra preserves the trivialities $X = \emptyset$ or $Y = \emptyset$ and the carriers in both ways. Then, both **3.3** and **5.3** give us the same set of all bijections $g: A \mapsto\!\!\twoheadrightarrow B$. (Note also that the above proof (geometric $\Rightarrow$ description) holds even for such dimensions.)

To check the three transformations, note that all derivations preserve any dimension triviality, each of which implies singletons as in (33), that **3.3** and **4.5** give us the same $t = \{\langle U, V \rangle\}$ by **3.1** (A) and **4.0** and that $t(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E} = \{\boldsymbol{i}_A\}$, by **5.3** and (geometric $\Rightarrow$ description). This also ensures that the two descriptions corresponds to their two transformations. *Q.E.D.*

6.1 Corollaries.

(A) *A description is a renaming iff the former base is the converse base,* $U = V'$.

(B) *When the former base and the converse one are co-indexed, $X = Y$, the converse base is a base,* $\boldsymbol{r}'_{V'} : \mathcal{E} \twoheadrightarrow A^X$.

(C) *A geometric description preserves flocks in both ways:* $a \in \Phi'_L$ *iff* $g(a) \in \Phi''_{t(L)}$ *for all* $a \in A$ *and* $L : X \to A$.

(D) *A matrix transformation induces a flock inclusion isomorphism, namely it preserves inclusion among flocks in both ways: for all* $L, M : X \to A$, $\Phi'_L \subseteq \Phi'_M$ *iff* $\Phi''_{t(L)} \subseteq \Phi''_{t(M)}$.

Proofs. (A) (If) Trivial for $X = Y = \emptyset$. Otherwise, assume that $U = g^{-1} \cdot V = V' : X \to A$. Then, by (34) and (16) $t(M)(x) = g(\chi_{g^{-1}(V(x))}(M)) = g(\chi_{U(x)}(M)) = g(M(x)) = (g \cdot M)(x)$ for all $M : X \to A$ and $x \in X$. Hence, (35) holds.

(Only if) From **6.0**, **4.5** and (35) $V = t(U) = n \cdot U$. This implies $U = n^{-1} \cdot V = V'$, since $n : A \twoheadrightarrow B$.

(B) Consider the function $\mathbf{B}_{g^{-1}} : B^X \to A^X$ such that $\mathbf{B}_{g^{-1}}(M) = g^{-1} \cdot M$ for all $M : X \to B$. From $g^{-1} : B \twoheadrightarrow A$ we easily get $\mathbf{B}_{g^{-1}} : B^X \twoheadrightarrow A^X$. Then, for all $e \in \mathcal{E}$, $\boldsymbol{r}_{V'}(e) = e \cdot g^{-1} \cdot V = g^{-1} \cdot (g \cdot e \cdot g^{-1}) \cdot V = g^{-1} \cdot \boldsymbol{r}''_V(\widetilde{g}_e) = \mathbf{B}_{g^{-1}}(\boldsymbol{r}''_V(\widetilde{g}_e)) = (\mathbf{B}_{g^{-1}} \cdot \boldsymbol{r}''_V \cdot \widetilde{g})(e)$, namely $\boldsymbol{r}'_{V'} = \mathbf{B}_{g^{-1}} \cdot \boldsymbol{r}''_V \cdot \widetilde{g} : \mathcal{E} \twoheadrightarrow A^X$ as in (0), since by **6.0** it is a composition of bijections also because of (13) and **5.2**.

(C) Let $a \in \Phi'_L$, namely $a = \chi_c(L)$ for some $c \in \Phi'_U$. Then, by **6.0** and (38) $g(a) = g(\chi_c(L)) = \xi_{g(c)}(t(L))$, namely by **5.4** $g(a) \in \Phi''_{t(L)}$. Clearly, we can reverse this implication by **3.4** (D) (Symmetry).

(D) (Only if) By (C) we can merely show that $g(a) \in \Phi''_{t(M)}$ for all $a \in \Phi'_L$. Since $\Phi'_L \subseteq \Phi'_M$, $a \in \Phi'_M$ and by (C) $g(a) \in \Phi''_{t(M)}$ for all such a's. (If) By symmetry: by (C) the premise becomes $g(a) \in \Phi''_{t(M)}$ for all $a \in \Phi'_L$, while the conclusion $a \in \Phi'_M$ again follows from (C) for all such a's. *Q.E.D.*

6.2 Example. Within vector spaces descriptions share some properties of the semi-linear transformations, which we will recall in **6.5** (A). Here, we recall that a *projectivity* is an inclusion isomorphism $p : \mathcal{S} \twoheadrightarrow \mathcal{T}$ between the sets of subspaces $\mathcal{S}$ and $\mathcal{T}$ of two vector spaces,

$$A' \subseteq A'' \text{ iff } p(A') \subseteq p(A'') , \quad \text{for all } A', A'' \in \mathcal{S} . \tag{58}$$

Let A and B respectively denote the carriers of the vector spaces. Then, we say that a bijection $g : A \twoheadrightarrow B$ *induces* p, when p is the corresponding restriction of the image function of g,

$$p(A') = \{g(a) \mid a \in A'\} , \quad \text{for all } A' \in \mathcal{S} \subseteq PA . \tag{59}$$

We prove that, *when the Menger systems or analytic monoids come from vector spaces, any description* $g : A \twoheadrightarrow B$ *induces a projectivity.*

Proof. Consider the vector-space flocks, defined as in **I.1** of [0], that are not the whole space. By the lemma in **VII.7** ibid. such proper flocks are all and only our flocks with respect to its Menger system (its vector times matrix multiplication), since a vector space has one dimension only. By the recalled definition the proper subspaces are all and only the flocks containing **0**. Therefore, given χ and ξ, we can define an injection $p:\mathcal{S}\Vdash\!\!\rightarrow PB$ from $g:A\Vdash\!\!\rightarrow B$ by (59) and also get $p:\mathcal{S}\Vdash\!\!\twoheadrightarrow\mathcal{T}$.

In fact, by **6.1** (C) $p(A') = \Phi''_{t(L)}$ for all $L:X \to A$ such that $A' = \Phi'_L$ and for all $A' \in \mathcal{S} \smallsetminus \{A\}$. The flock $\Phi''_{t(L)}$ must contain **0**, because by **6.0** and **4.8** (A) g commutes with the two null endomorphisms, the only constant valued ones in vector spaces. ($A' = A$ is a trivial case.) Conversely, by symmetry for every $B' \in \mathcal{T} \smallsetminus \{B\}$ we get an $M:Y \to B$ with $B' = \Phi''_M$ and an L with $M = t(L)$, such that $B' = p(A')$ for some $A' \in \mathcal{S} \smallsetminus \{A\}$ by **3.1** (B).

Finally, we get (58) by restricting **6.1** (D) to $\mathcal{S}$. *Q.E.D.*

An immediate corollary of this statement is that *within vector spaces descriptions preserve subspace dimensions*, since projectivities do.

6.3 Definitions. Let two algebras with our bases U and V define the Menger systems χ and ξ respectively. We say that a bijection $\varsigma':A\Vdash\!\!\twoheadrightarrow B$ is a *Segre description* between our Menger systems or analytic monoids or based algebras, when ς' is a centralizer bijection, as in **4.8** (A), that preserves the reference flocks and there is a surjection $\varsigma'':F\twoheadrightarrow G$ between scalars such that

$$\varsigma'(\chi_a(S)) = \xi_{\varsigma'(a)}(\varsigma''(S)) \quad \text{for all } a \in A \text{ and } S \in F\ . \tag{60}$$

The requirement $\varsigma'':F\twoheadrightarrow G$ is equivalent to merely require a relation $\rho \subseteq F \times G$ with $\{e \mid \langle e,f\rangle \in \rho\} = F$ and $\{f \mid \langle e,f\rangle \in \rho\} = G$ that contains ς''. Anyway, as the next proof will show, ς' and the bases define ς'' as an isomorphism between scalar monoids and we call the latter the *scalar isomorphism.*

Such descriptions involve both centralizer notions: the bijection one in **4.8** (A) and the sub-monoid one in **2.6** (A) and **2.7** (B). Note the definition symmetry: ς' is a Segre description between χ and ξ with ρ iff ς'^{-1} is a Segre description between ξ and χ with ρ^{-1}.

6.4 Theorem. *Any description is a Segre description and conversely.*

Proof. (description $\Rightarrow$ Segre) Take $\varsigma'' = t \cdot \boldsymbol{i}_F$ that by **6.0** and **4.7** (A) is an isomorphism between scalar monoids. Then, (60) with $\varsigma' = g$ is a restriction of (38). Hence, any description (which preserves the reference flocks) is a Segre description, because by **6.0** and **4.8** (A) $\varsigma' = g$ is the required centralizer.

(Segre $\Rightarrow$ description) The same triviality cases for a geometric description in **5.3** also occur for a Segre description. In fact, in that proof we only have to disregard dilatation indicators outside the reference flocks. Then, our statement is obvious and we assume $X, Y \neq \emptyset$.

Any Segre description with non trivial dimensions is a geometric description, because a centralizer bijection $\varsigma' = g$ preserves all endomorphisms, $\widetilde{\varsigma'}: \mathcal{E} \twoheadrightarrow \mathcal{F}$, as we observed in the proof (analytic $\Rightarrow$ geometric) of **6.0**, and because (60) and the reference flock preservation imply the full preservation of dilatations, as we are going to show.

In fact, for each $e \in \Delta$ we have $S = e \cdot U \in F$, $\varsigma''(S) \in G$ and the corresponding dilatation $f \in \Gamma$, $\boldsymbol{r}''_V(f) = \varsigma''(S)$, such that by (60), (14), and (15) $\varsigma'(e(a)) = f(\varsigma'(a))$ for all $a \in A$, namely $\varsigma' \cdot e = f \cdot \varsigma'$. Conversely, since $\varsigma'': F \twoheadrightarrow G$, given any $f \in \Gamma$ we have such an $e \in \Delta$. Hence, by **5.1** (A)

$$\widetilde{\varsigma'} \cdot \boldsymbol{i}_\Delta : \Delta \rightarrowtail\!\!\!\twoheadrightarrow \Gamma \quad \text{and} \quad \boldsymbol{r}''_V \cdot \widetilde{\varsigma'} \cdot \boldsymbol{i}_\Delta = \varsigma'' \cdot \boldsymbol{r}'_U \cdot \boldsymbol{i}_\Delta \ . \tag{61}$$

Moreover, for each $a \in A$ consider the endomorphism $h_a \in \mathcal{E}$ defined by (14) as $h_a(c) = \chi_c(\boldsymbol{k}_a)$ for all $c \in A$. Since ς' is a centralizer bijection, $\widetilde{\varsigma'}: \mathcal{E} \rightarrowtail\!\!\!\twoheadrightarrow \mathcal{F}$, for each a by (15) there is an endomorphism $\ell_a \in \mathcal{F}$ and a matrix $M_a : Y \to B$ such that $\varsigma'(\chi_c(\boldsymbol{k}_a)) = \varsigma'(h_a(c)) = \ell_a(\varsigma'(c)) = \xi_{\varsigma'(c)}(M_a)$ for all $c \in A$. Given any $y \in Y$, take $c = \varsigma'^{-1}(V_y)$ and get $\chi_c(\boldsymbol{k}_a) = a$ for each $a \in A$, since $c \in \Phi'_U$ by **2.1** (A) and the preservation of reference flocks. Then, by (60) and (19) $\varsigma'(a) = \varsigma'(\chi_c(\boldsymbol{k}_a)) = \xi_{V(y)}(M_a) = M_a(y)$ for each $a \in A$ and every $y \in Y$, namely by (9) $M_a = \boldsymbol{\kappa}_{\varsigma'(a)}$.

Since M_a is constant with respect to c, we got that $\varsigma'(\chi_c(\boldsymbol{k}_a)) = \xi_{\varsigma'(c)}(\boldsymbol{\kappa}_{\varsigma'(a)})$ for all $c \in A$. By the former of (61) this implies the full preservation of dilatations: $\widetilde{\varsigma'}(\chi_c \cdot \boldsymbol{k}) = \xi_{\varsigma'(c)} \cdot \boldsymbol{\kappa} \in \Gamma$ for all dilatation indicators c and conversely by the definition symmetry in **6.3**. Notice that the latter of (61), together with **2.7** (C), implies that the isomorphism $\varsigma'' = t \cdot \boldsymbol{i}_F$ is the only surjection between scalars satisfying (60). *Q.E.D.*

6.5 Missing proofs in Linear Algebra.

(A) The three main definitions of a description and their characterization in **6.0** define what means to say "descriptions are a general universal notion" from a theoretical point of view. Their Segre variant, as well as the one of **4.6** (A), serves more technical purposes: it shows how universal scalars work.

However, from a concrete point of view, also Segre descriptions serve to assess generality. In fact, (B) will show that they are a *formal* extension of the semi-linear transformations of vector spaces, whereas the general isomorphisms were not.

Conversely, one might like to check theoretically that the semi-linear transformations are the most general ones for vector spaces by proving that in vector spaces all descriptions have to be semi-linear transformations. Unfortunately, in spite that in (B) we will give a characterization, one cannot directly use it to prove this. In fact, we will show the lack of the proof of a renaming condition that Linear Algebra considered self-evident. Even the proofs of weaker conditions are missing.

Consider the general condition for semi-linear transformations as in **III.1** of [0]. It requires that, for any vector-space scalar $s \in \mathrm{F}$ and any vector $v \in A$,

$$\sigma'(sv) = \sigma''(s)\sigma'(v) \ , \tag{62}$$

where $\sigma' : A \rightarrowtail\!\!\twoheadrightarrow B$ denotes an isomorphism between the groups of the vector sums, $\sigma'' : \mathrm{F} \rightarrowtail\!\!\twoheadrightarrow \mathrm{G}$ an isomorphism between the fields concerned and, as usual for vector spaces, the two juxtapositions denote two *different* products of a scalar times a vector.

Semi-linear transformations relate two vector spaces regardless their reference frames. They can also relate their representations, after assuming $X = Y$, since they are projectivities and preserve dimensions, by setting

$$V = \sigma' \cdot U \ , \tag{63}$$

as we do for renamings by **6.1** (A). To focus this choice of Linear Algebra, we will call such transformations *semi-linear transformation between renamed reference frames.*

In **3.5** (A) $\mathrm{F} = \mathrm{G}$ was the set of complex numbers and $A = B$, while σ'' was conjugation and $\sigma' = g$ was vector conjugation. Notice that the only difference between (62) and (60) is the notation for the product scalar times vector, which in (62) is juxtaposition *both on the left and on the right.* Again, we have symmetry: σ' and σ'' define a semi-linear transformation iff σ'^{-1} and σ''^{-1} do.

Semi-linearity replaces the simple notion of a vector space as an algebra by a *split* one that concerns two algebraic structures, each of which undergoes its transformation. Moreover, the latter structure, the field as a division ring, is not a total (homogeneous) algebra.

Such a splitting does not occur in the universal algebras for our descriptions, though **2.7** (B)–(C) show that still an auxiliary (total) algebra always rises. It only occurs in the transformations: $\varsigma' = g$ is a Segre description only when there also is some $\varsigma'' = t \cdot \boldsymbol{i}_F$ for a matrix transformation $t = T_g$.

No partial algebra occurs in a Segre description: our scalars in $F \subseteq A^X$ or $G \subseteq B^Y$ merely form Abelian monoids as in **2.9** (B). Neither sums (of scalars or vectors) nor their distributivities are needed, as scalars analytically represent certain endomorphisms.

Within Linear Algebra, the assumption (63) is not completely specified, because in $V = t(U)$ one should define what t is. Here, on the contrary, we have such t's by (34). Then, through the above characterization **6.4** we can specify (63) as: *"if a Segre description is a semi-linear transformation, then it is between renamed reference frames"*.

Yet, a possible proof of such a statement will not fully prove the generality of semi-linear transformations. To save the abstract coordinate-free approach *within Linear Algebra, its birth niche*, we need a stronger statement: *"if a Segre description concerns vector spaces, then it is a renaming"*.

In fact, the latter proof could complete the next one in (B). Perhaps, we could get it by proving that *"the preservation of vector subspace dimensions, found in* **6.2**, *implies the renaming condition"*.

After counterexample **3.6** (A) and the uniqueness of **3.1** (C) the generality of renaming is untenable. Besides, in Linear Algebra, even the proof of the logical independence of the renaming condition is missing and one cannot get it as a new axiom. Once linear transformations (isomorphisms) were discarded, keeping their renaming feature needs some explanation.

(B) We prove that *every semi-linear transformation between renamed reference frames is a Segre description between the two corresponding based vector spaces. Conversely, whenever a Segre description is a renaming between two based vector spaces with dimensions greater than 1, it is a semi-linear transformation between the corresponding renamed reference frames.*

Proof. Since $U : X \to A$ is a vector-space base, we have its coordinating function $\mathrm{c}_U : A \Vdash\!\!\rightarrow \mathrm{F}^X$, such that, for each $v \in A$, $v = \sum_x (\mathrm{c}_U(v))_x U_x$ and we denote the coordinate $(\mathrm{c}_U(v))_x$ by $v[x]$. Given $V : X \to B$, we define $(\mathrm{c}_V(v))_x$ likewise. Then, by (62) and (63) $\sigma'(v) = \sum_x \sigma'(v[x]U_x) = \sum_x \sigma''(v[x])\sigma'(U_x) = \sum_x \sigma''(v[x])V_x$, since σ' preserves (finite) sums, namely

$$(\mathrm{c}_V(\sigma'(v)))_x = \sigma''(v[x]) \ , \quad \text{for all } x \in X \ . \tag{64}$$

σ' is a centralizer bijection, because for each $e \in \mathcal{E}$ by (14) and (62) there is $L : X \to A$ such that $\sigma'(e(v)) = \sigma'(\chi_v(L)) = \sigma'(\sum_x v[x]L(x)) = = \sum_x \sigma'(v[x]L(x)) = \sum_x \sigma''(v[x])\sigma'(L(x)) = \sum_x (\mathrm{c}_V(\sigma'(v)))_x(\sigma' \cdot L)(x)) = = \xi_{\sigma'(v)}(M) = f(\sigma'(v))$ for all $v \in A$, where $M = \sigma' \cdot L : X \to B$ by (15) represents $f \in \mathcal{F}$. Conversely, since $\sigma' : A \Vdash\!\!\twoheadrightarrow B$, given any such an f and M, there is L and $e \in \mathcal{E}$ such that $\sigma' \cdot e = f \cdot \sigma'$ as required by (53).

It preserves the reference flocks, because any $v \in \Phi'_U$ has the form $v = \sum_x c_x U_x$, where $c_x \in \mathrm{F}$ for all $x \in X$ and $\sum_x c_x = 1$, as remarked in **2.0**. Hence, by (64) $\sigma'(v) = \sum_x \sigma''(c_x)V_x$, where again $\sum_x \sigma''(c_x) = \sigma''(\sum_x c_x) = 1$, since σ'' is a field isomorphism. Conversely, any $v' \in \Phi''_V$ is a $v' = \sigma'(v)$ and we can reverse these implications to get $v \in \Phi'_U$. Then, we can set $\varsigma' = \sigma'$.

Finally, let us define $\varsigma'' : F \Vdash\!\!\twoheadrightarrow G$. Consider the diagonal matrix isomorphism of **2.8**, $D : \mathrm{F} \Vdash\!\!\twoheadrightarrow F$. Likewise, consider $D' : \mathrm{G} \Vdash\!\!\twoheadrightarrow G$. Then, $\varsigma'' = D' \cdot \sigma'' \cdot D^{-1}$ is a surjection onto G as required. By (62) it also satisfies (60), because for every

$S \in F$ there is $s \in \mathrm{F}$ with $S = D_s$ such that by (29) $\varsigma'(\chi_a(S)) = \sigma'(sa) = \sigma''(s)\sigma'(a) = \xi_{\sigma'(a)}(D'_{\sigma''(s)}) = \xi_{\sigma'(a)}(D'_{\sigma''(D^{-1}(S))}) = \xi_{\varsigma'(a)}(\varsigma''(S))$ for all $a \in A$, since (29) holds even in the latter Menger system.

(Conversely) As the dimensions are the same by **6.2**, we take $Y = X$. To prove that $\varsigma' : A \twoheadrightarrow B$ is an isomorphism between the groups of the vector sum, we define an endomorphism application that performs sums in our space of dimension greater than 1. Take two different $x, y \in X$, their "sum vector" $u = U_x + U_y$ and, for all $a, b \in A$, the matrices $L : X \to A$ and $M : X \to B$ such that $L(x) = a$, $M(x) = \varsigma'(a)$, $L(y) = b$, $M(y) = \varsigma'(b)$ and $L(z) = \mathbf{0}$, $M(z) = \mathbf{0}$ elsewhere.

Notice that the latter matrix comes from the former by ς'–images, $M = \varsigma' \cdot L$, because ς' commutes with the null endomorphisms. Hence, by **6.4** and the renaming assumption $M = T_{\varsigma'}(L)$.

Then, $\varsigma'(a + b) = \varsigma'(\chi_u(L)) = \xi_{\varsigma'(u)}(\varsigma' \cdot L) = \xi_{\varsigma'(u)}(M)$ by **6.4** and (36). Consider two coordinates of $\varsigma'(u)$ with respect to V: $c_x = \mathrm{c}_V(\varsigma'(u)))_x$ and $c_y = \mathrm{c}_V(\varsigma'(u)))_y$. Then, $\varsigma'(a + b) = c_x M_x + c_y M_y$. In particular this holds for $a = U_x$ and $b = \mathbf{0}$ as well as for $a = \mathbf{0}$ and $b = U_y$. Therefore, $c_x = c_y = 1$ by **6.1** (A) and

$$\varsigma'(a + b) = \varsigma'(a) + \varsigma'(b) \quad \text{, for all } a, b \in A \; . \tag{65}$$

Also, $\sigma' = \varsigma'$ satisfies (62) with $\sigma'' = D'^{-1} \cdot \varsigma'' \cdot D$ as above. In fact, by (29), (60) and (29) $\sigma'(sv) = \varsigma'(\chi_v(D_s)) = \xi_{\varsigma'(v)}(\varsigma''(D_s)) = \xi_{\varsigma'(v)}(D'(D'^{-1}(\varsigma''(D_s)))) = \xi_{\varsigma'(v)}(D'_{\sigma''(s)}) = \sigma''(s)\sigma'(v)$ for all $v \in A$ and $s \in \mathrm{F}$.

Such a $\sigma'' : \mathrm{F} \twoheadrightarrow \mathrm{G}$ also is a field isomorphism. In fact, it preserves multiplications, because, in addition to ς'', both D and D'^{-1} trivially do. Moreover, once one has defined sums on F and G by columnwise vector sums, one also easily finds both that such sums are (universal) scalars and that D and D'^{-1} preserve sums. Hence, to prove that σ'' preserve sums, one only has to prove that ς'', or also $\varsigma'' \cdot D$, does.

By the bijective representation of scalars as dilatations in **2.7** (C) this is to prove that, given any $s', s'' \in \mathrm{F}$, for all $b = \varsigma'(a) \in B$, $\xi_b(\varsigma''(D(s' + s''))) = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$, namely by (60) that $\varsigma'(\chi_a(D(s' + s''))) = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$. In fact, the field distributivity gets $\varsigma'(\chi_a(D(s' + s''))) = \varsigma'((s' + s'')a) = \varsigma'(s'a + s''a) = \varsigma'(s'a) + \varsigma'(s''a) = \varsigma'(\chi_a(D(s'))) + \varsigma'(\chi_a(D(s''))) = \xi_b(\varsigma''(D(s'))) + \xi_b(\varsigma''(D(s''))) = \varsigma''(D(s'))b + \varsigma''(D(s''))b = (\varsigma''(D(s')) + \varsigma''(D(s'')))b = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$ by (29), (65), (29) again, (60) and (29) twice again. *Q.E.D.*

6.6 Analytic spaces. Counterexample **3.6** denies universal generality to the abstract representation-free approach of Algebra. Then, somebody might look for a *subclass* of algebras, where such an abstract Algebra continues to work.

This subclass is the one where all descriptions are renamings. Safely, it deserves further studies. Yet, its lack of proper descriptions does not save all conventional wisdom. The following example, a sort of converse of **3.6** (B), shakes the very notion of an algebra as a single concrete object.

Consider an algebra that has singleton bases as well as bases of $n > 1$ reference elements. The one of B. Jónsson & A. Tarski in [8] is the simplest. Given a singleton base $U : 1 \to A$, *any matrix transformation from U has to reach other singleton bases only.* In fact, by characterization **4.6** (A) it has to be a bijection $t : A^1 \rightarrowtail\!\!\!\!\twoheadrightarrow A^Y$ that retypes $\boldsymbol{K}$ as in **1.6**. This is impossible unless Y too is singleton, as shown in **1.7** (B).

Therefore, by **6.1** (A) any description from U is a renaming, because the reference flocks are singleton. Hence, we stay in our comfortable subclass, e.g. now we can say to have 1 as a dimension. The trouble is that this one-dimensioned space is not the whole algebra, because its self-descriptions cannot reach invariant properties that need larger bases to be formalized. *In spite of the common carrier and operations* it cannot sense higher dimensions.

(Besides, one should not leave this one-dimensioned space to go back to the the abstract algebra: abstract representation-free properties can lack invariance. See **6.11** of [13] for some well-known abstract properties of Universal Algebra that can fail after performing mere automorphisms.)

Then, from a concrete point of view, this algebra is not a single mathematical object, but a superposition of this one-dimensioned space with other space(s). Our analytic monoids or Menger systems, which formalize such "analytic spaces" together with the equivalence of **3.4** (D) or the category of **3.7**, can peer at them. Yet, nothing can melt them to get such a thing as a free "algebra without the choice of a base".

One might well dismiss our based algebra as a "paradoxical" one. Yet, some preliminary results in [20], mentioned in [17], show that its one-dimensioned space provides both a word catenation monoid and a binary tree algebra with a natural common extension and hint that it can improve one of the best computer memory organization so far known. Its other spaces, as well as the dimensionless space of **3.6** (A), could likely provide us new methods for memory addressing.

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