Uniform relational frameworks for modal inferences*

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Abstract. The design and the implementation of variable-free deductive frameworks (ultimately based on Tarski's arithmetic of dyadic relations) for aggregate theories, strongly rely on the availability of suitable pairing notions.
Thanks to a set-theoretical treatment of modalities, we show that even in the case of modal propositional logics, an appropriate pairing notion can be introduced and exploited in order to devise an alternative approach to modal deduction.

Key words: *Modal logic*, *relational systems*, *translation methods*.

Introduction

In this paper we focus on a technical issue which plays a crucial role in algebraic formalization of set theory. The kind of formalization we have in mind sprouts from the historical line of work presented in the monograph [17]; but our contribution is much more specific, as it links with previous work on set-theoretic renderings of non-classical logics, and such renderings only presuppose very weak axioms concerning sets. It has been shown, in fact (cf. the *box-as-powerset*, in brief $\Box$-as-$\mathcal{P}$, translation of [4, 1]), that even a very weak theory can offer adequate means for expressing the semantics of modal systems of propositional logic, for investigating the issue of their first-order representability, and for automating inferences in non-classical contexts (cf. [10, pp. 478–481]). Unlike previous studies, where suitable weak set theories were formalized within ordinary quantified calculus (in one case, namely [14], an inferential system *à la* Rasiowa-Sikorski was developed), here we provide a variable-free, equational version of the target set-theoretic language. To compensate for the lack of variables and quantifiers, the equality construct will be used for comparing expressions designating global relations between sets rather than for comparing set-expressions.

Providing a suitable notion of *pairing* is the central issue of the proposal we put forward; in fact, pairing will be a key tool for obtaining, with minimal axiomatic commitment, the desired equational support for the said set-rendering

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of modal logics specified *à la* Hilbert. Such a tool enables use of the Maddux-Monk-Tarski algorithm (cf. [17]) for restating any first-order sentence in three variables; so it paves the way to the equational treatment of non-classical logics. As the present paper will illustrate through various examples stemming from the general approach discussed in [2], the proposed pairing device can be built into a specialized variant of the $\Box$-as-$\mathcal{P}$ translation so as to drive it into a completely equational setting.

An opportunity for improvements would have been missed if we had simply proposed a montage of the $\Box$-as-$\mathcal{P}$ translation with Maddux' translation. To mention one thing, the first-order sentences resulting from $\Box$-as-$\mathcal{P}$ constitute a rather narrow sublanguage of first-order logic, whose 3-variable translation can easily be obtained by more direct means than by the general method. Moreover, in view of the special purpose of our translation, the first-order (set-theoretic) equivalent of a modal proposition can be left understood, and the two logical phases of the overall translation can be fused into a single process. The role of Maddux' translation, in this process, can be superseded by the graph-based approach developed in [2], whose techniques, however, are conservative and do not guarantee success in all circumstances; therefore, suitable extensions of that approach had to be devised to take full advantage of the special context of our discourse, where most of the relations entering into play are functions, and where conjugated projections (left and right inverses of the pairing function) are available.

As for the pairing function, none seems to be available under the very weak assumptions of the target set theory as originally proposed in [4, 1]. We must either add an explicit pair axiom, or design a suitable variant of the $\mathcal{P}$ operation (cf. Sec.1): in either case, a well-known device due to Kuratowski (whereby ordered pairs can be encoded in terms of nested unordered pairs) can be adopted.

This paper is 'twin', in a sense, of [14]. Both develop a theory-based approach, whose difference relative to a more typical calculus-based approach will be sketched in Sec.3. Suffice it to say, for the time being, that, typically, one assumes the universe of discourse to consist exclusively of the possible worlds of a frame; here, instead, the universe of discourse results from the amalgamation of all Kripkean frames and hence encompasses worlds and frames together (as well as other less relevant entities), all uniformly viewed as 'sets'. Under one important aspect this paper and [14] diverge, though: the Rasiowa-Sikorski system adopted there is particularly handy and human-oriented, whereas the equational kind of reasoning to which modal logics are reduced in this paper is intended to be mainly oriented to machine formula-crunching.

1 Background axiomatic aggregate theory

Consider the axiomatic theory Ω whose postulates are

$$\begin{array}{r} \forall y\,\forall x\,\exists n\,\forall v\big(\ v \in n \ \leftrightarrow \quad (v \in x \ \&\ v \in y)\ \big)\,, \\ \forall y\,\forall x\,\exists d\,\forall v\big(\ v \in d \ \leftrightarrow \ \neg\,(v \in x \leftrightarrow v \in y)\ \big)\,, \\ \forall x\,\exists p\,\forall v\big(\ v \in p \ \leftrightarrow \quad (\forall u \in v)(u \in x)\ \big)\,. \end{array}$$

By resorting to Skolem operators $\cap$, $\triangle$, and $\mathcal{P}$, and to the syntactic abbreviation

$$v \subseteq x \ \leftrightarrow_{\mathrm{Def}} \ (\forall\, u \in v)(u \in x)\,,$$

and leaving the quantifiers $\forall\, v$, $\forall\, x$, $\forall\, y$ tacit, we can recast the above axioms more perspicuously as follows:

$$\begin{array}{lcl} v \in x \cap y & \leftrightarrow & (v \in x \quad \& \quad v \in y)\,, \\ v \in x \triangle y & \leftrightarrow & (v \in x \quad \leftrightarrow \quad \neg\, v \in y)\,, \\ v \in \mathcal{P}(x) & \leftrightarrow & v \subseteq x\,. \end{array}$$

One can view Ω as being an extremely weak theory of 'aggregates' which becomes a genuine set theory only after appropriate postulates, such as the *extensionality* axiom

$$(x \subseteq y \ \& \ y \subseteq x) \ \rightarrow \ x = y$$

and the *(unordered) pair* axiom

$$v \in \{x, y\} \ \leftrightarrow \ (v = x \vee v = y)\,,$$

are added to it. On the other hand Ω is known to be, already as it stands, an ideal target first-order theory into which to translate mono-modal systems of propositional logic uniformly (cf. [3, Chapter 12]): in the translation, the converse $\ni$ of membership acts as a relation which includes immediate accessibility between possible worlds; accordingly, $\cap$ and $\triangle$ play the role of the classical connectives $\&$, $\oplus$—conjunction and exclusive disjunction—, and $\mathcal{P}$ corresponds to the necessity operator $\Box$.

From the standpoint of this '$\Box$-*as*-$\mathcal{P}$' translation, the weakness of Ω is a virtue rather than a defect. As a matter of fact, if the extensionality axiom were included in Ω, this would set an undesirable limitation to its usability in the study of non-classical logics; and a similar objection can be raised against postulates entailing the well-foundedness of membership. Certain enrichments of Ω with new postulates, e.g. the addition of the pair axiom, do not jeopardize applicability of the $\Box$-as-$\mathcal{P}$ translation method; nevertheless such enrichments appear to be unjustified unless they are shown to yield some technical—perhaps computational—advantages.

The important result summarized in this paragraph, established by Alfred Tarski over half century ago and later improved by J. Donald Monk and Roger Maddux (cf. [17, Chapter 4]), seems to favor the addition of the pair axiom to Ω. An effective procedure exists for reducing each sentence of the language underlying any first-order theory of membership which includes the pair axiom to an equivalent sentence involving three variables only; furthermore, this procedure enables global translation of such a theory into a purely equational extension of the *arithmetic of dyadic relations*, which is an algebraic theory owning an "almost embarrassingly rich structure" [12]. Thanks to such a translation, we might gain better service from today's theorem provers run in autonomous mode: in our own experience, these generally demonstrate higher performances when confronted with purely equational theories than with theories which more fully exploit the symbolic first-order apparatus [8, 9]. By translating modal axioms and theses into an equational variant of Ω, we could hence best benefit of current proof technology.

We will propose below an even less committing way of reducing (a variant Ω' of) Ω to the arithmetic of relations, taking advantage of the fact that the above cited Tarski-Maddux' result holds also for theories where an analogue of the pair axiom, of the form

$$\forall y\, \forall x\, \exists q\, \forall v\big(\, v \;\mathsf{in}\; q \;\leftrightarrow\; (v = x \vee v = y)\,\big)\,,$$

can be derived from the axioms. The only requirement, in regard to this, is that "v in q" be a formula which involves three variables altogether and has v and q as its sole free variables. To achieve our translation purpose, we just have to retouch the one axiom which characterizes the 'powerset' operator $\mathcal{P}$ so that it behaves more naturally when the extensionality axiom is missing. Our proposed replacement for the third axiom of Ω is the sentence

$$\forall x\, \exists p\, \forall v\big(\, v \in p \;\leftrightarrow\; (v = x \vee v \subset x)\,\big)\,,$$

where

$$v \subset x \;\;\leftrightarrow_{\mathrm{Def}}\; \neg\,(v \subseteq x \rightarrow x \subseteq v)$$

(that is, $v \subset x$ holds if and only if every element of v belongs to x whereas x has some element not belonging to v). Under this revised axiom, even without extensionality axiom, it is clear that exactly one p, let us call it $\tilde{\mathcal{P}}(x)$, corresponds to each x so that the elements of p are precisely x and all of its strict subsets $v \subset x$. Likewise, to any q there corresponds at most one a such that q max a holds, where

$$q \;\mathsf{max}\; a \;\;\leftrightarrow_{\mathrm{Def}}\; \big(a \in q \;\&\; q \subseteq \tilde{\mathcal{P}}(a)\big)\,;$$

but, unlike $\tilde{\mathcal{P}}$ which is total, max is a *partial* function of its first operand.

In our revised version Ω' of Ω, one can conceive an analogue of the unordered pair $\{a, b\}$ to be $\tilde{\mathcal{P}}(a)$ when $a = b$ and to have the same elements as $\tilde{\mathcal{P}}(a) \triangle \tilde{\mathcal{P}}(b)$ when $a \neq b$. (Out of such 'unordered pairs', one can proceed to construct 'ordered pairs' analogous to Kazimierz Kuratowski's pairs $\langle a, b\rangle =_{\mathrm{Def}} \big\{\{a, b\}, \{a, a\}\big\}$, and these will behave as desired, but let us avoid a discussion on this point taking it for granted [6].) With this rationale in mind, we can characterize as follows a 'pseudo-membership' which meets the formal analogue seen above of the pair axiom:

$$b \;\mathsf{in}\; q \;\;\leftrightarrow_{\mathrm{Def}}\; \Big(b \in q \;\&\; \big(\neg\,\exists\, d \in q\big)\big(b \subset d\big)\Big) \vee \exists\, a\Big(q \;\mathsf{max}\; a \;\&\; b \subset a \;\&\; \forall\, d\Big(d \in q \leftrightarrow \big(d \in \tilde{\mathcal{P}}(a) \;\&\; \neg\, d \in \tilde{\mathcal{P}}(b)\big)\Big)\Big)\,.$$

To see that in can be specified in three variables, it suffices to observe that since

$\ni =_{\rm Def} \in^{\smile}$	$\supseteq =_{\rm Def} \overline{\not\ni \in}$
$\supset =_{\rm Def} \supseteq \cdot \ni\!\notin$	$\subset =_{\rm Def} \supset^{\smile}$
$\tilde{\mathcal{P}} =_{\rm Def} \in - \supset\!\notin - (\bar{\iota} \cdot \not\supset) \in$	$\mu =_{\rm Def} \tilde{\mathcal{P}} \supseteq \cdot \in$
$\mathsf{in} =_{\rm Def} \Big(\in - \subset\in \Big) \sqcup \Big(\subset \mu - \tilde{\mathcal{P}} \ni\in - \mathbb{1} \,(\in - \in \tilde{\mathcal{P}}^{\smile} \mu) - \tilde{\mathcal{P}} \not\ni (\in \tilde{\mathcal{P}}^{\smile} \mu - \in) \Big)$	
$\gamma =_{\rm Def} (\mathsf{in} - \bar{\iota}\, \mathsf{in})\, \mathsf{in}$	$\mathsf{syq}(P,\, Q) =_{\rm Def} \overline{P^{\smile}\ \overline{Q}} \cdot \overline{\overline{P}^{\smile}\ Q}$
$\lambda =_{\rm Def} \gamma - \bar{\iota}\, \gamma$	$\varrho =_{\rm Def} \mathsf{in}\, \mathsf{in} - \bar{\iota}\, (\mathsf{in}\, \mathsf{in} - \lambda)$
$\tilde{\mathcal{P}}\, \mathbb{1} = \mathbb{1}$	$\lambda\, \varrho^{\smile} = \mathbb{1}$
$\mathbb{1} = \mathbb{1}\ \mathsf{syq}(\in, \in \lambda \cdot \in \varrho)$	$\mathbb{1}\ \mathsf{syq}(\in, \in \lambda + \in \varrho) = \mathbb{1}$

Fig. 1. Specification of Ω' in the arithmetic of dyadic relations

max is single-valued, the *definiens* of the predicate in can be rewritten as follows:

$$\Big(b \in q \ \&\ \big(\neg \exists\, d \in q\big)\big(b \subset d\big)\Big) \ \vee\ \Big(\exists\, d\, (q \ \mathsf{max}\ d \ \&\ b \subset d) \\ \&\ \forall\, d \Big(d \in q \leftrightarrow \Big(\big(\neg\, d \in \tilde{\mathcal{P}}(b)\big) \ \&\ \exists\, b\big(q \ \mathsf{max}\ b \ \&\ d \in \tilde{\mathcal{P}}(b)\big)\Big)\Big)\Big).$$

Given the above definition of in, let us consider the following pair of relations:

$$a\, \lambda\, q \ \leftrightarrow_{\rm Def}\ a\, \gamma\, q \ \&\ \forall\, z\, (z\, \gamma\, q \rightarrow z = a),$$
$$b\, \varrho\, q \ \leftrightarrow_{\rm Def}\ \exists\, w\, (b \ \mathsf{in}\ w \ \&\ w \ \mathsf{in}\ q) \ \&\ \forall\, z\, \big(\exists\, v\, (z \ \mathsf{in}\ v \ \&\ v \ \mathsf{in}\ q) \rightarrow z = b \vee z\, \lambda\, q\big),$$

where $x\, \gamma\, y \ \leftrightarrow_{\rm Def}\ \exists\, z\, \big(x \ \mathsf{in}\ z \ \&\ z \ \mathsf{in}\ y \ \&\ \forall\, w\, (w \ \mathsf{in}\ z \rightarrow w = x)\big)$.
Since in is expressed in three variables, it is easy to verify that both λ and ϱ are defined in terms of 3-variable sentences. Notice that, thanks to their definitions, both λ and ϱ are functions of their second operands. According to [17] (see also [6] and Sec.5 below), the availability of such a pair of relations allows one to recast the pairing axiom in three variables, as follows:

$$\forall\, x\, \forall\, y\, \exists\, q\, (x\, \lambda\, q \ \&\ y\, \varrho\, q).$$

The equational rendering of this axiom will be part of the equational specification of Ω', as shown in Figure 1 (where an equational rendering of λ and ϱ is also shown).

At present we do not know whether Ω', without further axioms, can supersede Ω as target first-order theory into which modal propositional logics can be translated naturally. In fact, the proof of the theorem which states the adequacy of the $\Box$-as-$\mathcal{P}$ translation into Ω (cf. [4, Sec.3]) cannot be carried over in an obvious manner to our slightly altered context.

2 Arithmetic of homogeneous dyadic relations

We got *in medias res*.[4] The algebraic specification of Ω' just reached calls for a quick flash back into our own variant of the algebraic form of logic historically

[4] Directly to the heart of the tale.

developed by Charles Sanders Peirce, Ernst Schröder, and Alfred Tarski [17], to recall a few.

Our intended *universe of discourse* is a collection $\Re$ of dyadic relations over a non-null domain $\mathcal{U}$. We assume that the *top* relation $\bigcup\Re$, and the *diagonal* relation, consisting of all pairs $\langle u,u\rangle$ with u in $\mathcal{U}$, belong to this universe, which is also closed under the *intersection*, *symmetric difference*, *composition*, and *conversion* operations. Within our symbolic algebraic system, the constants $\mathbb{1}$ and ι designate the top and the diagonal relation; moreover $\cdot$ and $+$ designate intersection and symmetric difference; composition is represented by simple juxtaposition; and conversion by the monadic operator $^{\smile}$. These operations are interpreted as follows, where for any relational expression R we are indicating by $R^{\Im}$ the relation (over $\mathcal{U}$) designated by R:

- $P^{\smile}$ designates the relation consisting of all pairs $\langle v,u\rangle$ with $\langle u,v\rangle$ in $P^{\Im}$;
- $P\,Q$ designates the relation consisting of all pairs $\langle u,w\rangle$ such that there is at least one v for which $\langle u,v\rangle$ and $\langle v,w\rangle$ belong to $P^{\Im}$ and to $Q^{\Im}$, respectively;
- $P{\cdot}Q$ designates the relation consisting of all pairs $\langle u,v\rangle$ which simultaneously belong to $P^{\Im}$ and to $Q^{\Im}$;
- $P{+}Q$ designates the relation consisting of all pairs $\langle u,v\rangle$ which belong either to $P^{\Im}$ or to $Q^{\Im}$ but do not belong to both of them.

Designations for customary operations over relations can be introduced through shorthand definitions, e.g. as follows:

$$\overline{Q} =_{\mathrm{Def}} Q{+}\mathbb{1}, \qquad \emptyset =_{\mathrm{Def}} \overline{\mathbb{1}},$$
$$P{-}Q =_{\mathrm{Def}} P{\cdot}\overline{Q}, \qquad P{\sqcup}Q =_{\mathrm{Def}} \overline{\overline{Q}{-}P}.$$

We will sometimes exploit alternative notation for the complement operation and for equations of a special kind:

$$\not{Q} =_{\mathrm{Def}} \overline{Q}, \qquad P{\sqsubseteq}Q \;\leftrightarrow_{\mathrm{Def}}\; P{-}Q{=}\emptyset.$$

Further abbreviations can be introduced at will, as shown for example in Figure 1 by the introduction of the *symmetric quotient* operation syq (cf. [16, pp. 19–20, 71ff]). As concerns *priorities*, we assign to $^{\smile}$, composition, $\cdot, +, -$, and $\sqcup$ decreasing cohesive powers; moreover, all dyadic constructs, namely $\cdot, +, -, \sqcup$, and composition, are assumed to associate to the left.

Figure 2 displays an axiomatic presentation of the *arithmetic of relations*, which also encompasses the standard equational inference rules. We regard these axioms as *logical* ones, because they form, in a sense (together with the inference rules), a *calculus* on top of which one can build purely equational theories, such as the Ω' theory introduced above. Specific *theories* will talk about special relations—the one designated by $\in$, in the case of Ω'—, which they constrain to comply with *proper* axioms—e.g. the ones shown in Figure 1. A theory characterizes peculiar domains endowed with special relations. For example, the axioms of Ω' imply that the domain $\mathcal{U}$ is infinite and has a considerable amount of structure. This actually reflects our willingness to impose enough structure on $\mathcal{U}$ that it can be regarded as an amalgamation of all Kripke frames upon which the '*possible-world*' semantics of modal propositional logics is based.

$P \cdot Q = Q \cdot P$	$P \cdot (Q + R) + P \cdot Q = P \cdot R$
$(P \cdot Q) \cdot R = P \cdot (Q \cdot R)$	$(P + Q) + R = P + (Q + R)$
$\mathbb{1} \cdot P = P$	$\iota \; P = P$
$(P \; Q) \; R = P \; (Q \; R)$	$(P \sqcup Q) \, R = (Q \, R \sqcup P \, R)$
$(P \cdot Q)^{\smile} = Q^{\smile} \cdot P^{\smile}$	$(P \, Q)^{\smile} = Q^{\smile} \, P^{\smile}$
$P^{\smile\smile} = P$	$Q \cdot \big((Q \; P + \mathbb{1}) \; P^{\smile}\big) = \emptyset$

Fig. 2. Logical axioms of the arithmetic of dyadic relations

3 Direct relational translations of modal logics

A vast literature exists on how to translate non-classical propositional logics into first-order predicate calculus and how to exploit such translations for automated non-classical reasoning, cf. e.g. [18].

Consider, for example, the following translation mapping (essentially the one proposed in [15]), which associates a relational expression $t(\varphi)$ with each modal propositional sentence φ :

- $t(p_i) \;=_{\mathrm{Def}}\; p'_i \;\; \mathbb{1}$, where p'_i is a relational variable uniquely corresponding to p_i , for every propositional variable p_i;
- $t(\neg\,\psi) \;=_{\mathrm{Def}}\; \overline{t(\psi)}$, for every propositional sentence ψ ;
- $t(\psi \;\&\; \chi) \;=_{\mathrm{Def}}\; t(\psi) \cdot t(\chi)$, for all propositional sentences ψ, χ ;
- $t(\Diamond\,\psi) \;=_{\mathrm{Def}}\; \mathsf{r} \;\; t(\psi)$, where r is a constant designating the *accessibility relation* between possible worlds, for every propositional sentence ψ .

Of course the connectives $\rightarrow, \vee, \leftrightarrow, \oplus, \Box$ can be handled similarly, via reductions to $\neg$, $\&$, $\Diamond$. It is also plain that $t(\varphi)$ designates a *right-ideal* relation Φ; namely, one which satisfies the equation $\Phi \; \mathbb{1} = \Phi$.

Assume that we have been able to capture the semantics of a specific logic $\mathbb{L}$ by means of a system of relational equations $\mathcal{E}(\mathbb{L})$ *involving* r *alone*: then we can, in place of the problem of establishing whether or not $\models_{\mathbb{L}} \vartheta$ (where ϑ is any modal formula), address the equivalent problem of establishing whether or not $\mathcal{E}(\mathbb{L}) \vdash t(\vartheta){=}\mathbb{1}$. One can see the condition $\mathcal{E}(\mathbb{L}) \vdash t(\vartheta){=}\mathbb{1}$ as a statement referring to first-order predicate calculus, because we can re-express the (relatively unusual) relational constructs it involves in terms of individual variables and quantifiers. Indeed, we can rewrite $P{=}\mathbb{1}$ as $\forall x \forall y (\, x \, P \, y \,)$, $x P \cdot Q y$ as $x \, P \, y \;\; \& \;\; x \, Q \, y$, $xPQy$ as $\exists\, z (x \, P \, z \;\; \& \;\; z \, Q \, y)$, etc.

This approach is viable for a wide spectrum of non-classical logics, including some which are directly characterized in terms of semantic constraints which the accessibility relation is subject to, for example *extensionality*

$$\mathsf{syq}(\mathsf{r},\, \mathsf{r}) \sqsubseteq \iota$$

(a property of contact relations used in spatial reasoning, cf. [5]). On the other hand, there exist modal logics very simply characterizable via Hilbert-like axioms which *do not* admit a first-order correspondent, one such being the single-axiom Löb logic

$$\Box(\Box p \rightarrow p) \rightarrow \Box p \,.$$

For such defective logics a correspondent can always be found in *second-order* predicate calculus, which unfortunately is not complete (no matter what recursive set of logical axioms is chosen, cf. [13]).

An alternative possibility, which we will discuss in the ongoing, is to take a first-order *theory* (as opposed to the mere calculus) as the target formalism for the translation. A most natural choice, when $\mathbb{L}$ is given by a finite conjunction α of Hilbert-like axioms, is to refer to the arithmetic $\mathcal{RA}$ of relations (see Figure 2): we can then take $\mathcal{E}(\mathbb{L})$ to be $t(\alpha){=}\mathbb{1}$, and try to see whether $\models_{\mathbb{L}} \vartheta$ by checking whether $\mathcal{RA}$ & $\big(t(\alpha){=}\mathbb{1}\big) \vdash t(\vartheta){=}\mathbb{1}$. This approach would lack completeness too (since $\mathcal{RA}$ does not axiomatize the whole variety of representable relation algebras), despite being sound, efficient (as automatic theorem-proving in purely equational contexts is faster than for full first-order logic), and able to provide answers in common cases.

From its very origin, the theory Ω was devised with the aim that it should retain enough of the power of second-order predicate calculus to make recourse to second order useless, while ensuring completeness. In its original form, however, Ω was not immediately amenable to an equational extension of $\mathcal{RA}$; hence it did not enable emulation of non-classical reasoning in the purely equational part of the arithmetic of relations; now, thanks to the availability of conjugated projections, it will.

4 Modal logics and the arithmetic of relations

Let us recall from [4], [1] (see also [3, Chapter 12]) that there is a translation $\varphi \mapsto \varphi^\star$ of modal propositional sentences into set-terms of Ω which enjoys the following properties:

- A sentence schema $\varphi \equiv \varphi[p_1, \ldots, p_n]$ built from n distinct propositional meta-variables p_i becomes a term $\varphi^\star \equiv \varphi^\star[f, x_1, \ldots, x_n]$ involving $n+1$ distinct set-variables, one of which, f, is meant to represent a generic frame.
- If $\varphi^\star \equiv \varphi^\star[f, \boldsymbol{x}]$ and $\psi^\star \equiv \psi^\star[f, \boldsymbol{y}]$ result from propositional schemata φ, ψ, then the biimplication

$$\psi \models_{\mathsf{K}} \varphi \Leftrightarrow \Omega \vdash \forall f \big(\mathsf{is\_frame}(f) \wedge \forall \boldsymbol{y}(f \subseteq \psi^\star) \rightarrow \forall \boldsymbol{x}(f \subseteq \varphi^\star)\big)$$

  holds, where K is the minimal modal logic and $\mathsf{is\_frame}(\,\cdot\,)$ characterizes those elements of the intended domain $\mathcal{U}$ which represent Kripke frames.

Hence, by combining this translation with a proof system for Ω, one achieves a proof system which can be exploited to semi-decide any finitely axiomatized mono-modal propositional logic (and even, in favorable cases, to decide it). We would like to now tune the same translation for our relational theory Ω', with the additional advantage that the target formalism would be a purely equational one in the case at hand; however, due to the difficulty mentioned at the end of Sec.1, in the rest of this paper we replace Ω' by the (equational) theory Ω'' consisting of the axioms of Ω' plus the axiom

$$\mathcal{P}\,\mathbb{1} = \mathbb{1}, \quad \text{where} \quad \mathcal{P} =_{\text{Def}} \mathsf{syq}(\subseteq, \in) \quad \text{and} \quad \subseteq =_{\text{Def}} \overline{\ni\notin}\,.$$

We begin by specifying the monadic relation $\mathsf{is\_frame}$, bearing in mind that since $\ni$ must act, when restricted to a frame, as the accessibility relation between worlds in that frame, a frame f must be a transitive set, i.e., it must satisfy $f \subseteq \mathcal{P}(f)$. This characterization turns out to be adequate to our purposes:

$$\mathsf{is\_frame} =_{\text{Def}} \iota \cdot \mathcal{P} \supseteq \;.$$

Then we must specify operations on $\mathcal{U}$ that correspond to the propositional constructs $\vee$, $\&$, and $\Box$. Here the rationale is that the term $\varphi[f, \boldsymbol{x}]$ into which one translates a propositional schema $\varphi^\star[\boldsymbol{p}]$ represents the collection of all worlds (in the frame f) where φ holds. We have announced already in Sec.1 that the natural counterparts of $\Box$ and $\&$ will be $\mathcal{P}$ and $\cap$: analogously, $\cup$ and $\setminus$ will act as counterparts of $\vee$ and $\not\rightarrow$, but since we have not introduced any of $\cap$, $\cup$, and $\setminus$ explicitly in Figure 1, we can fill this gap now by putting[5]

$$\cap =_{\text{Def}} \mathsf{syq}(\in, \in\lambda \cdot \in\varrho)\,, \qquad \cup =_{\text{Def}} \mathsf{syq}(\in, \in\lambda \sqcup \in\varrho)\,,$$
$$\setminus =_{\text{Def}} \mathsf{syq}(\in, \in\lambda - \in\varrho)\,.$$

In the language underlying the Skolemized first-order version of Ω'', it is quite straightforward to define the mapping $\varphi[\boldsymbol{p}] \mapsto \varphi^\star[f, \boldsymbol{x}]$ of sentence schemata into terms:

$$p_i^\star =_{\text{Def}} x_i\,, \qquad (\neg\,\psi)^\star =_{\text{Def}} f \setminus \psi^\star\,,$$
$$(\psi \rightarrow \chi)^\star =_{\text{Def}} (\neg\,\psi)^\star \cup \chi^\star\,, \qquad (\Box\,\psi)^\star =_{\text{Def}} \mathcal{P}(\psi^\star)\,.$$

(of course we can also handle the connectives $\&$, $\vee$, $\leftrightarrow$, $\oplus$, $\Diamond$ via reductions to $\neg$, $\rightarrow$, $\Box$). This is called the *$\Box$-as-$\mathcal{P}$ translation.* Assuming that we manage to specify the relation $f \not\subseteq \varphi^\star[f, \boldsymbol{x}]$ between a frame f and a tuple $\boldsymbol{x}$ by means of a relational expression $\widehat{\varphi}$, then the translation of $\Psi \vdash_{\mathsf{K}} \varphi$ (where Ψ stands for a finite collection of sentence schemata) into a derivability problem regarding the (quantifier-free, purely equational) relational version of Ω'' will be

$$\Omega'' \;\vdash\; \mathsf{is\_frame} \cdot \widehat{\varphi}\,\mathbb{1} \sqsubseteq \widehat{\&\Psi}\;\mathbb{1}\,.$$

5 Compliance of a frame with a modal schema

How can we specify algebraically the relation (complement of the $\widehat{\varphi}$ just introduced) which holds between a frame f and a tuple $\boldsymbol{x}$ when $f \subseteq \varphi^\star[f, \boldsymbol{x}]$? A

[5] Needless to say, we must distinguish between $\cdot, +, -, \sqcup, \sqsubseteq$, which operate on relations, and corresponding constructs $\cap, \triangle, \setminus, \cup, \subseteq$, which operate on $\mathcal{U}$.

crucial observation is that within the relational version of Ω'' one can derive equations

$$L^{\smile}\, L \sqsubseteq \iota\,, \qquad R^{\smile}\, R \sqsubseteq \iota\,, \qquad L^{\smile}\, R = \mathbb{1}\,, \qquad \iota \sqsubseteq L\, \mathbb{1}\, R^{\smile}\,,$$

for suitably chosen relational expressions L, R. For example, we can take

$$L =_{\mathrm{Def}} \lambda^{\smile} \sqcup (\iota - \lambda^{\smile}\, \mathbb{1})\,, \qquad R =_{\mathrm{Def}} \varrho^{\smile} \sqcup (\iota - \varrho^{\smile}\, \mathbb{1})\,.$$

This remark immediately discloses the possibility of translating the full-fledged language of first-order Ω'' into the language of the arithmetic of relations: we can rely, for that, on a fundamental algorithm due to R. Maddux and explained in [17, Sec. 4.4]. Notice that the very fact that we can speak of *tuples* belonging to $\mathcal{U}$ relies on the availability of L, R. As a matter of fact, we can view each element a of $\mathcal{U}$ as a pair $\langle b, c\rangle$ whose components fulfill $aL^{\Im}b$ and $aR^{\Im}c$; accordingly, since we can decompose c in the same fashion, any element of $\mathcal{U}$ encodes a tuple of any desired length. On the other hand, $L^{\smile}\, R = \mathbb{1}$ implies that we can assemble a pair $\langle b, c\rangle$ from any two elements b, c of $\mathcal{U}$, and hence we can form an n-tuple with given components $b_1, \ldots, b_n$ for any finite number n.

We are not so much interested in translating the full first-order language into the equational one as we are in translating formulas $f \subseteq \varphi^{\star}[f, \boldsymbol{x}]$: these are, in fact, the ones we really need in order to carry out the translation of any sentence of modal propositional logic into relational Ω''. To this end, we can resort to a more direct graph-based translation approach explained in [2], which has been implemented by means of a tool for algebraic graph transformation named AGG (acronym for 'Attribute Graph Grammars'), developed at the TU, Berlin [11, 7].

Figure 3 displays the graph representation of $f \subseteq \varphi^{\star}[f, \boldsymbol{x}]$, when φ is one of the following modal sentences (x, y correspond to p and q, respectively):

S4: $\Box p \;\rightarrow\; \Box\Box p$;
Löb: $\Box(\Box p \;\rightarrow\; p) \;\rightarrow\; \Box p$;
K: $\Box(p \;\rightarrow\; q) \;\rightarrow\; (\Box p \;\rightarrow\; \Box q)$.

When fed with one of these three graphs, the thinning algorithm of [2, p. 455] terminates in a dead end. Figure 4 displays two of the resulting irreducible graphs, obscuring orientation and labels of the edges. Figures 5, 6, and 7 show, on a couple of examples, how to take advantage of the available 'projections' $\lambda^{\smile}$, $\varrho^{\smile}$ to bring an irreducible graph to a form whence the thinning algorithm can resume its job and terminate successfully, with the desired relational translation. (Notice that, the last graph of Figure 7 can be further reduced as done for the example in Figure 5.)

The key point is that, thanks to the availability of a pair of projections, $\lambda^{\smile}$ and $\varrho^{\smile}$, the thinning algorithm can be enriched with a new graph-rewriting rule:

PAIR-ENCODING rule. Let L and R be a pair of conjugated projections. Let ν' and ν'' be two nodes in a graph. Then introduce a new (bound) node ν together with two labelled edges $[\nu, L, \nu']$ and $[\nu, R, \nu'']$.

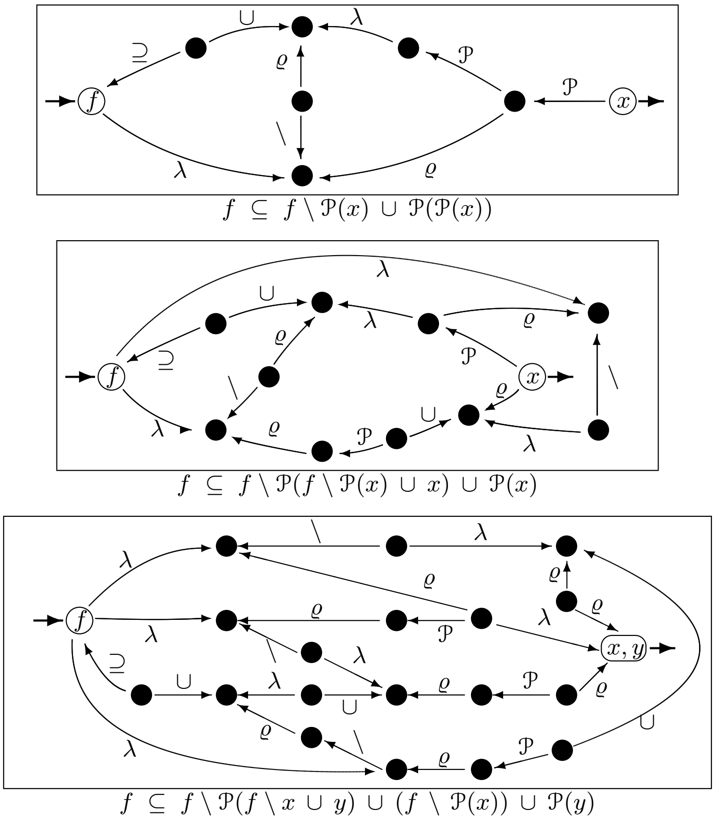


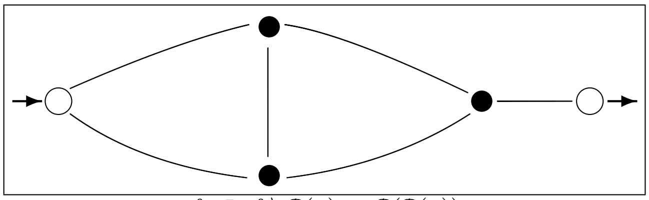
Fig. 3. Graph renderings of **S4**, **Löb**, and **K**

Such a rule is extremely general and, in principle, there is no restriction on its applicability. Clearly, it is convenient to exploit this rule within a strategy that relates its application to the firing of other rules of the thinning algorithm. In particular, since L and R are single-valued, the following rewriting-rule (cf. [2]) becomes crucial in order to bring to end the graph-rewriting process:

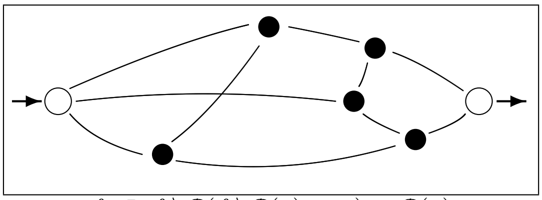
FUNCTIONALITY rule. Let $[\nu, P, \nu']$ be a labelled edge such that P is single-valued and let $[\nu', Q, \nu'']$ be another edge, with $\nu \neq \nu''$. Then the edge $[\nu', Q, \nu'']$ is removed and the new edge $[\nu, P\,Q, \nu'']$ is added to the graph. (If the graph contains another edge between ν and ν'' labelled S, then a fusion is made and the new edge will be labelled $P\,Q{\cdot}S$.)

Conclusions

In this paper we moved the first step toward the realization of an alternative deductive framework for non-classical logics based on pure equational reasoning.



$f \subseteq f \setminus \mathcal{P}(x) \cup \mathcal{P}(\mathcal{P}(x))$



$f \subseteq f \setminus \mathcal{P}(f \setminus \mathcal{P}(x) \cup x) \cup \mathcal{P}(x)$

Fig. 4. Irreducible graphs resulting from **S4** and from **Löb** (cf. Figure 3)

We demonstrated how to profitably combine the experiences issuing from two apparently weakly-related streams of research: We (re-)forged a set-theoretical approach to the deduction problem in modal theories, within a purely equational deductive framework designed for set-reasoning and ultimately based on Tarski's relation algebra. To this aim we exploited the very same techniques developed for equational re-engineering of various aggregate theories [8, 6]. At the same time, this paper presents a new significant improvement of the translation technique proposed in [2] in order to compile first-order sentences into the algebra of relations.

Experimental activities with a state-of-the-art theorem-prover such as Otter [8] seem to indicate that equational formulations of aggregate theories can favorably compete with more conventional first-order formulations. Through the proposed equational rendering of a Ω'', thanks to the $\Box$-as-$\mathcal{P}$ translation, modal propositional logics are amenable to the jurisdiction such equational automated reasoning methods. Such a result emphasizes, once more, the expressive and deductive power of Tarski's algebraic theory of relations.

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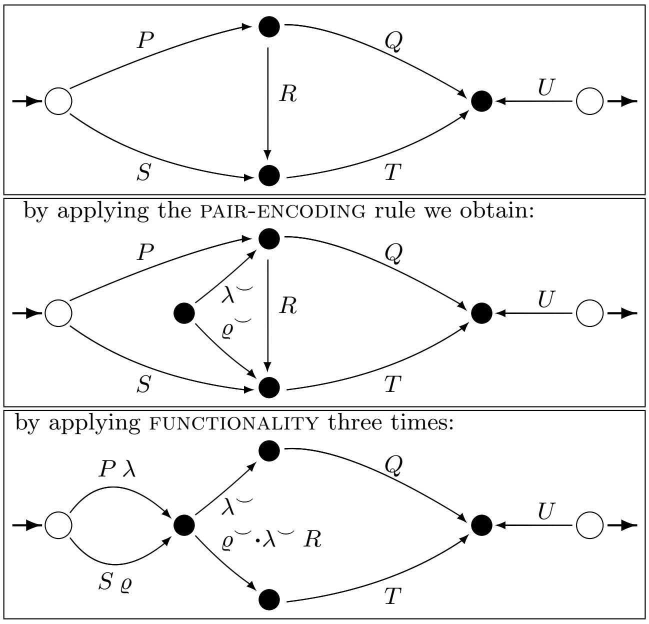

Fig. 5. Rehabilitation of an irreducible graph (cf. the first graph in Figure 4)

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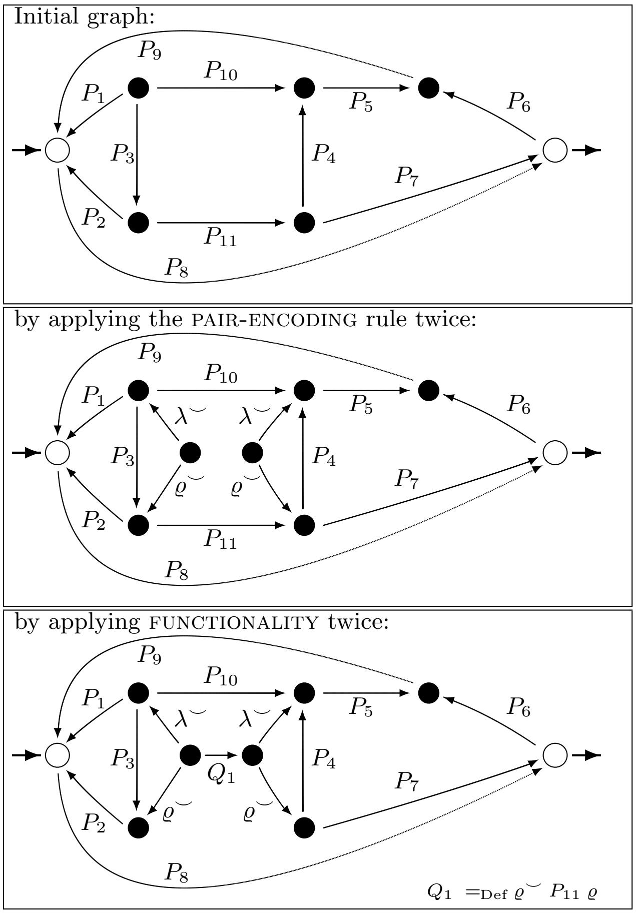

Fig. 6. Rehabilitation of an irreducible graph (continued in Figure 7)

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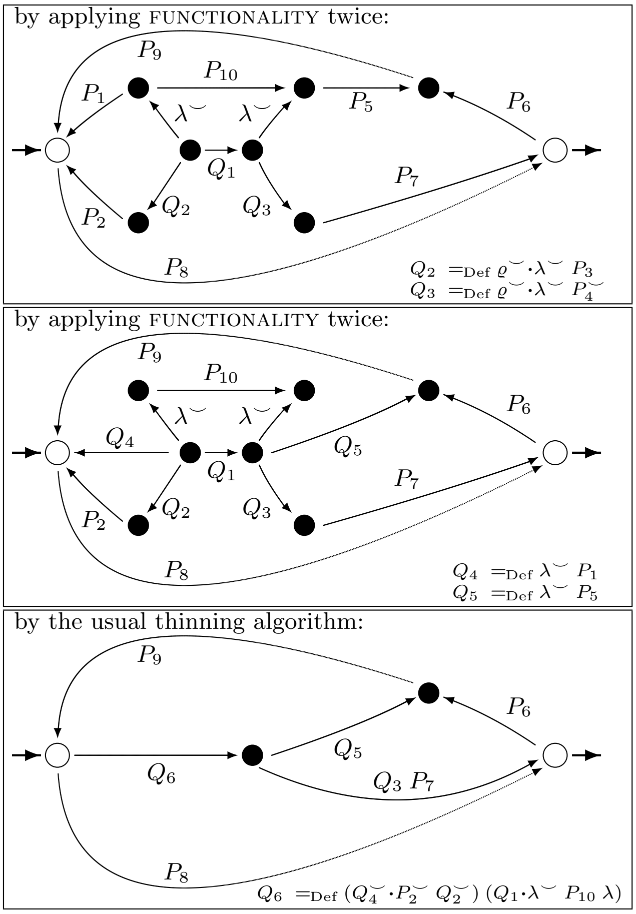

Fig. 7. Rehabilitation of an irreducible graph

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