Convexity Recognition of the Union of Polyhedra

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Currenty with esmertec AG http://www.esmertec.com

Joint Work with

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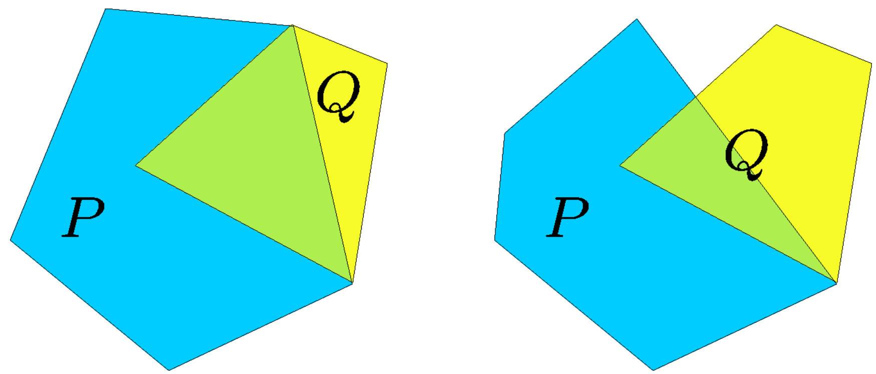
Union of Polyhedra



Problem: Given two convex polyhedra P,Q in $\mathbb{R}^d$:

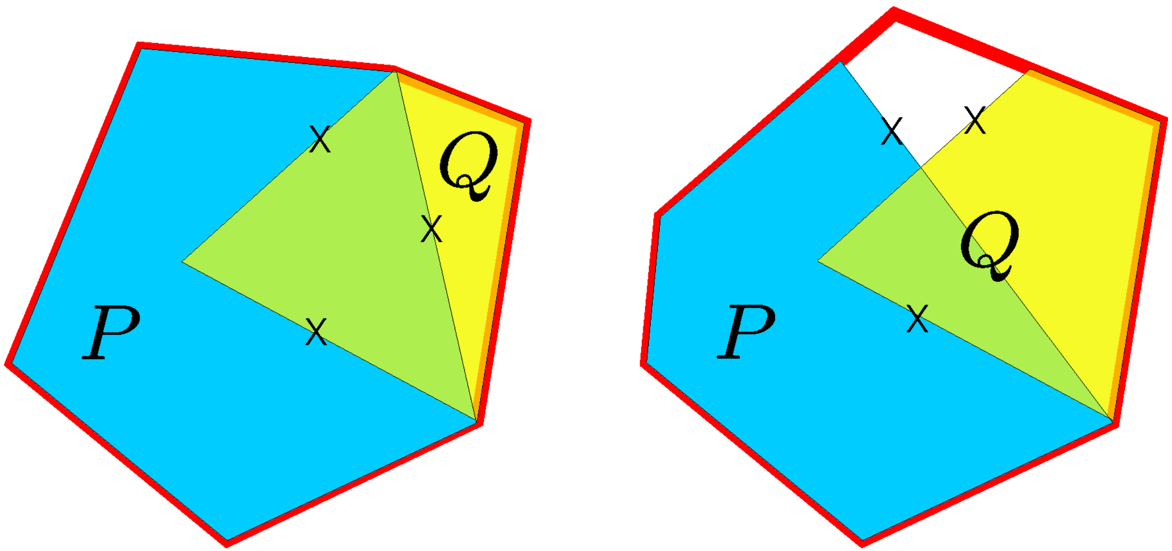
- Recognize if $P \cup Q$ is convex
- If yes, find a minimal representation for $P \cup Q$

Three natural cases:

1. P,Q, are in H-representation $P = \{x : Ax \leq \alpha\}$, $Q = \{x : Bx \leq \beta\}$
2. P,Q, are in V-representation $P = \text{conv}(V) + \text{cone}(R)$, $Q = \text{conv}(W) + \text{cone}(S)$
3. P,Q, are in VH-representation

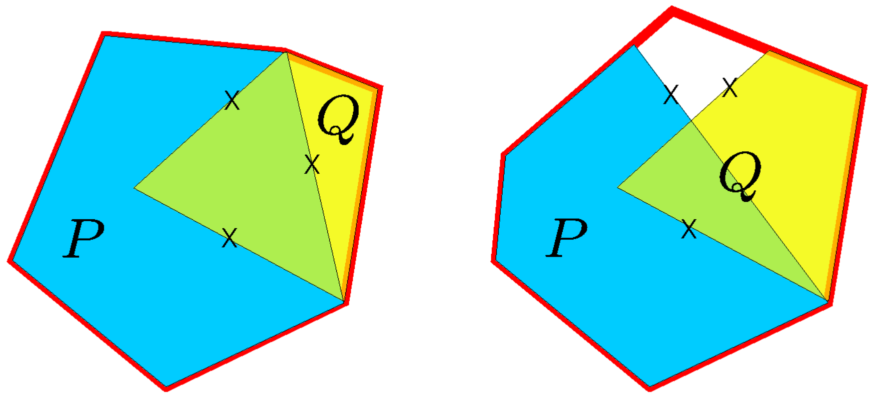
Envelope of Polyhedra

$$P = \{x : Ax \leq \alpha\}, \qquad Q = \{x : Bx \leq \beta\}$$



$$\text{env}(P, Q) \triangleq \{x : \bar{A}x \leq \bar{\alpha}, \bar{B}x \leq \bar{\beta}\}$$

Key Theorem for H-Polyhedra



Theorem 1 *$P \cup Q$ is convex $\Leftrightarrow$ $P \cup Q = \text{env}(P, Q)$.*

Proof.

$\Leftarrow$ trivial (env is a convex object)

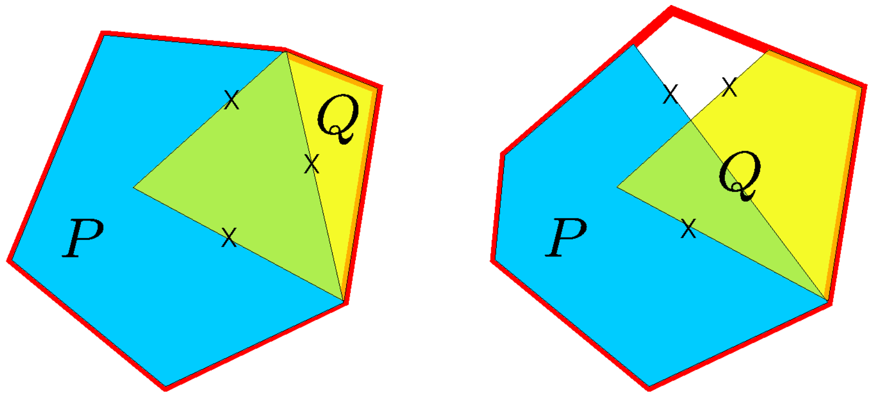
$\Rightarrow$ – Consider the H-representation of $P \cup Q$

– Clearly, $P \cup Q \subseteq \text{env}(P, Q)$

– Show that $P \cup Q \supseteq \text{env}(P, Q)$, by contradiction

- Holds for unbounded polyhedra

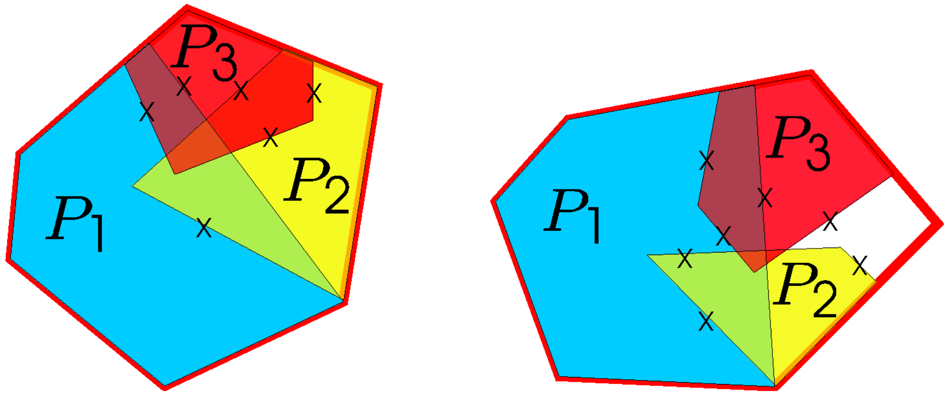
Key Theorem for H-Polyhedra



Proof.

- $P \cup Q = \text{env}(P, Q) \Rightarrow P \cup Q$ is convex trivial
- $P \cup Q$ is convex $\Rightarrow P \cup Q = \text{env}(P, Q)$
  - Let $K \triangleq P \cup Q$, $K \subseteq \text{env}(P, Q)$, show that $K \supseteq \text{env}(P, Q)$
  - assume by contradiction that a facet inequality $r'x \leq s$ of K is not in $\text{env}(P, Q)$
  - Let $H = \{x : r'x - s\}$, then $\dim(P \cap H) \leq d - 2$ and $\dim(Q \cap H) \leq d - 2$, $\dim(K \cap H) = d - 1$
  - $K \cap H = (P \cap H) \cup (Q \cap H)$, contradiction

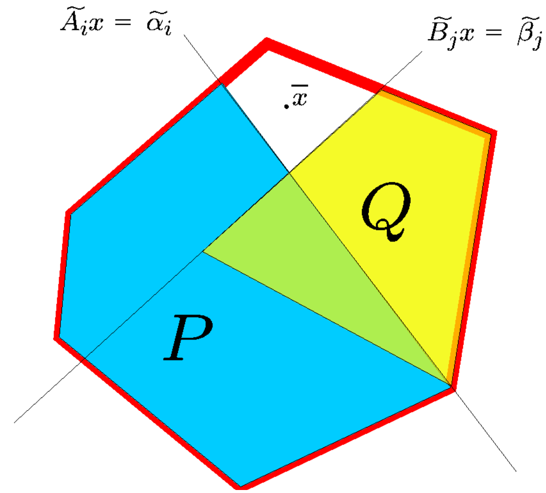
Generalization to k H-Polyhedra



- Theorem 1 generalizes to $k \geq 3$ H-polyhedra

$$\cup_{i=1}^{k} P_i \text{ is convex} \Leftrightarrow \text{env}(P_1, P_2, \ldots, P_k) = \cup_{i=1}^{k} P_i$$

Algorithm for H-Polyhedra



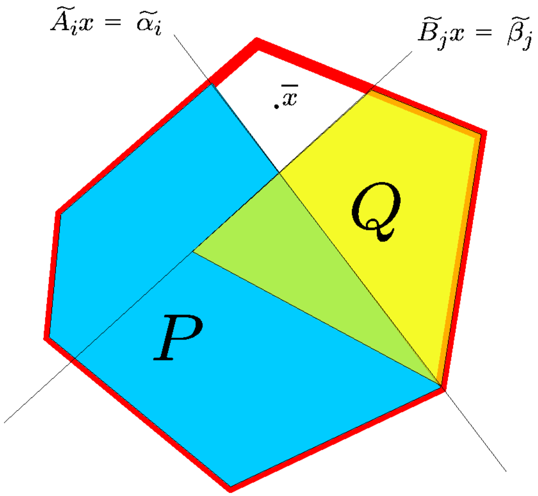
Main idea: Determine a point $\bar{x} \in \text{env}(P, Q)$, $\bar{x} \notin P$, $\bar{x} \notin Q$

Construct $\text{env}(P, Q) = \{x : \ Cx \leq \gamma\}$
let $\tilde{A}x \leq \tilde{\alpha}$, $\tilde{B}x \leq \tilde{\beta}$ be the set of removed constraints
Remove from $\text{env}(P, Q)$ possible duplicates
for each pair $\tilde{A}_i x \leq \tilde{\alpha}_i$, $\tilde{B}_j x \leq \tilde{\beta}_j$ **do**
  Determine ϵ^* by solving the linear program

$$\begin{aligned} \epsilon^* = \quad & \max_{(x,\epsilon)} && \epsilon \\ & \text{subj. to} && \tilde{A}_i x = \tilde{\alpha}_i + \epsilon \\ & && \tilde{B}_j x = \tilde{\beta}_j + \epsilon \\ & && Cx \leq \gamma \end{aligned}$$

  if $\epsilon^* > 0$, **stop**; **return False**;   /*$P \cup Q$ is nonconvex.*/
return $\text{env}(P, Q)$.   /*$P \cup Q$ is convex.*/

Algorithm for H-Polyhedra



$P = \{Ax \leq \alpha\}$, $Q = \{Bx \leq \beta\}$

Input: (A, α), (B, β) minimal H-representation

Output: Minimal H-representation of $P \cup Q$

Complexity: $O(m_1 m_2 \mathbf{lp}(d, m_1 + m_2))$

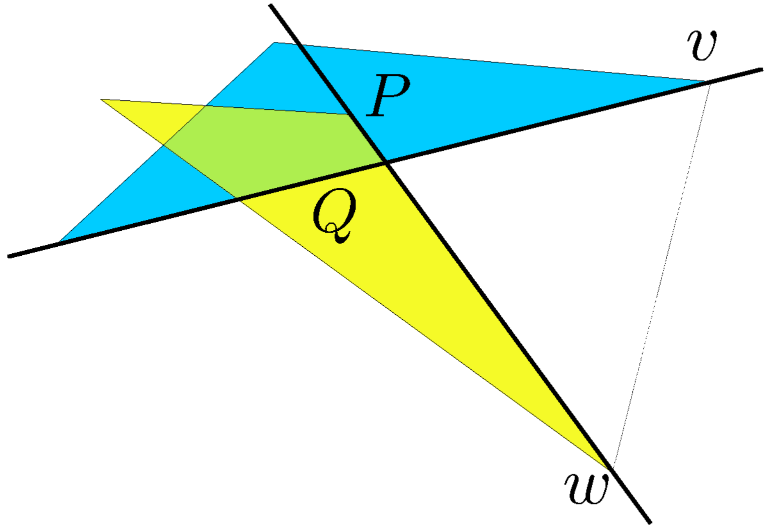
$m_1 =$ number of rows of A

$m_2 =$ number of rows of B

In general: $O(\prod_{i=1}^{k} m_i \mathbf{lp}(d, \sum_{i=1}^{k} m_i))$

Key Theorem for V-Polytopes

$P = \text{conv}(V), \qquad Q = \text{conv}(W)$



Theorem 2 *Let P, Q be polytopes with V-representation V and W, respectively. Then*

$$P \cup Q \text{ is convex} \Leftrightarrow [v, w] \subseteq P \cup Q,\ \forall v \in V,\ \forall w \in W.$$

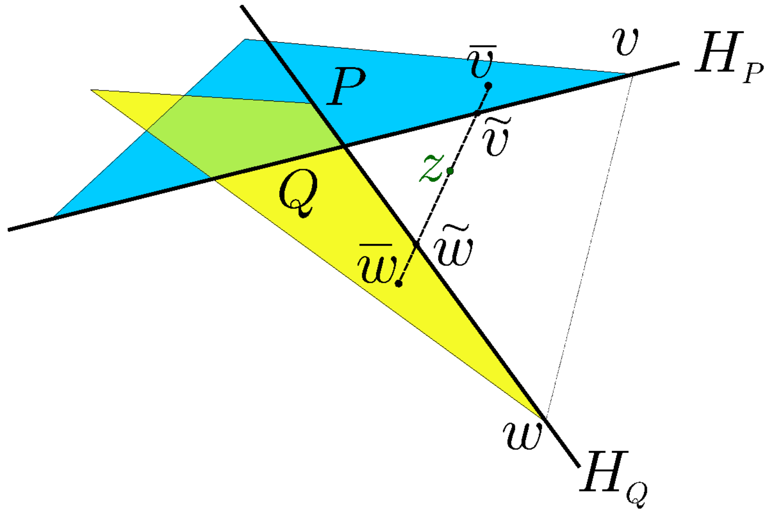
Moreover, a stronger characterization of convexity holds

$$\exists v \in V,\ w \in W\colon (v, w) \cap (P \cup Q) = \emptyset \Leftrightarrow P \cup Q \text{ is nonconvex.}$$

Generalizes to unbounded polyhedra by homogenization

Key Theorem for V-Polytopes

$$P = \text{conv}(V), \qquad Q = \text{conv}(W)$$



Proof.

$\Rightarrow$ $P \cup Q$ is convex $\Rightarrow [v, w] \subseteq P \cup Q, \; \forall v \in V, \; \forall w \in W$, trivial

$\Leftarrow$ $[v, w] \subseteq P \cup Q, \; \forall v \in V, \; \forall w \in W \Rightarrow P \cup Q$ is convex

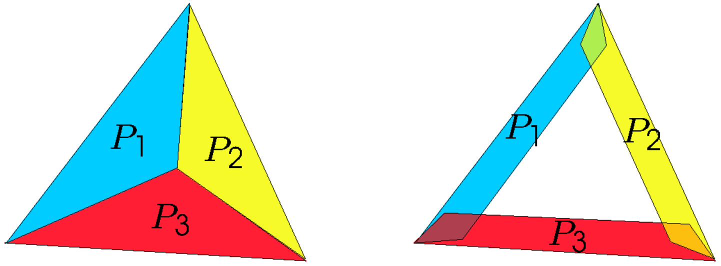
by contradiction assume $\exists z = (1-\gamma)\bar{v} + \gamma\bar{w}, 0 \leq \gamma \leq 1, \bar{v} \in P, \bar{w} \in Q$ s.t. $z \notin P \cup Q$

$\Rightarrow \exists \tilde{v}, \tilde{w}$ s.t. $(\tilde{v}, \tilde{w}) \subset [\bar{v}, \bar{w}], (\tilde{v}, \tilde{w}) \not\subseteq P \cup Q$, $\tilde{v} \in H_P, \bar{w} \in H_P^-$

$\Rightarrow \exists w \in W, w \in H_P^-$, similarly $\exists v \in V, v \in H_Q^-$

$\Rightarrow (v, w) \cap P \cup Q = \emptyset$, contradiction

Generalization to k V-Polyhedra



Open problem:

- Generalization to $k \geq 3$ V-polyhedra

Conjecture (false):

- Θ union of the vertices of P_1, P_2, P_3
- Check $\forall \theta_i, \theta_j \in \Theta$, $[\theta_i, \theta_j] \subset P_1 \cup P_2 \cup P_3$

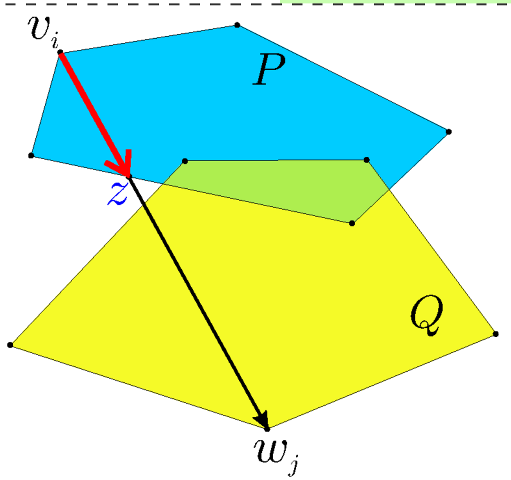
Alternative: (Carathodory) Consider the convex hull of $(d+1)$ vertices and check if it is contained in $P_1 \cup P_2 \cup P_3$

Theorem: The convex hull of k vertices is enough

Proof. By Carateodory's Theorem (Finschi, Torrisi)

Can we exploit this characterization in an algorithm?

Algorithm for V-Polyhedra (1)



Main idea: Ray shooting from v_i towards w_j, and check $z \in Q$

```
Remove vertices of P which are in Q, and vice-versa, and
let V̄, W̄ the sets of remaining vertices;
if no vertex has been removed, return False;     /*disjoint.*/
for each pair v_i ∈ V̄, w_j ∈ W̄ do
  Find the corresponding vector z ≜ v_i + λ*_0(w_j − v_i),
  where λ*_0 = max λ_0 s.t. v_i + λ_0(w_j − v_i) ∈ P
  Determine if z ∈ Q (via LFT)
  if z ∉ Q, return False;     /*P ∪ Q is nonconvex.*/
let X be the set of points in V ∩ W that are extreme in P ∪ Q;
return conv(V̄ ∪ W̄ ∪ X).
```

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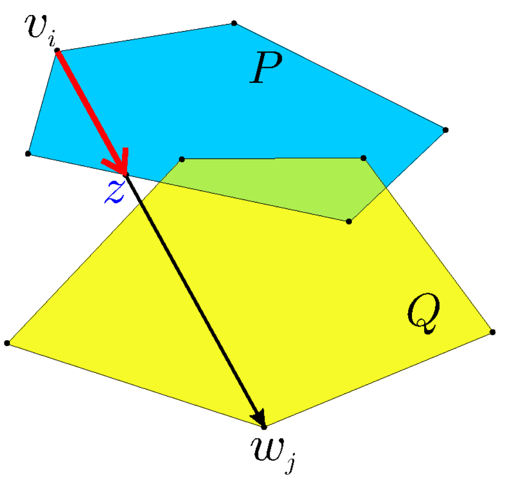
Algorithm for V-Polyhedra (1)



$P = \mathrm{conv}(V)$, $V = \{v_1, \ldots, v_{n_1}\}$

$Q = \mathrm{conv}(W)$, $W = \{w_1, \ldots, w_{n_2}\}$

Input: V, W minimal V-representation

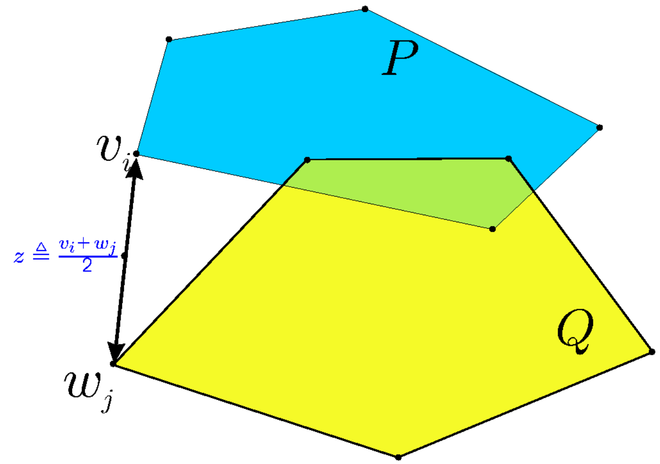
Output: Minimal V-representation of $P \cup Q$

Complexity: $O(n_1 n_2(\mathbf{lp}(d, n_1) + \mathbf{lp}(d, n_2)))$

$n_1 =$ number of vertices of P

$n_2 =$ number of vertices of Q

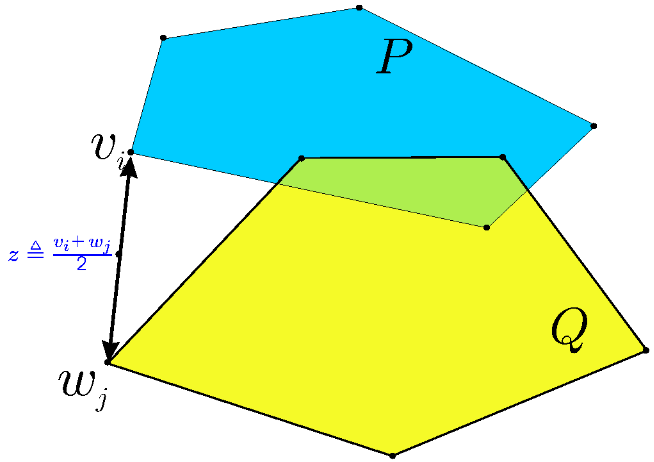
Algorithm for V-Polyhedra (2)



Main idea: Exploit the stronger converse result in Theorem 2, and try to find a segment $(v_i, w_j) \notin P \cup Q$ by checking the middle point $z = \frac{v_i+w_j}{2}$

```
Remove vertices of P which are in Q, and vice-versa, and
let V̄, W̄ the sets of remaining vertices;
if no vertex has been removed, return False;     /*disjoint*/
for each pair v_i ∈ V̄, w_i ∈ W̄ do
  let z ≜ (v_i + w_j)/2;
  Determine if z ∈ P ∪ Q (via LFT)
  if z ∉ P ∪ Q, return False;     /*P ∪ Q is not convex.*/
let X be the set of points in V ∩ W that are extreme in P ∪ Q;
return conv(V̄ ∪ W̄ ∪ X).
```

Algorithm for V-Polyhedra (2)



$P = \text{conv}(V)$, $V = \{v_1, \ldots, v_{n_1}\}$

$Q = \text{conv}(W)$, $W = \{w_1, \ldots, w_{n_2}\}$

Input: V, W minimal V-representation

Output: Minimal V-representation of $P \cup Q$

Complexity: $O(n_1 n_2(\mathbf{lp}(d, n_1) + \mathbf{lp}(d, n_2)))$

n_1 = number of vertices of P

n_2 = number of vertices of Q

Comparison:

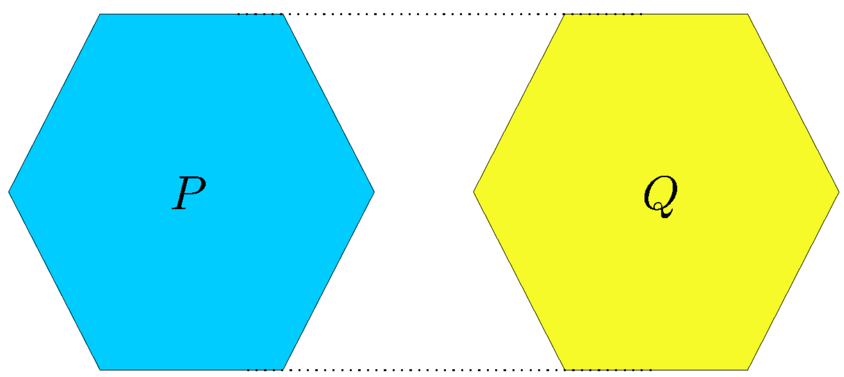
- Algorithm 1 might stop earlier if $P \cup Q$ is not convex

$\rightarrow$ performance depends on inputs P and Q

VH-Polytopes

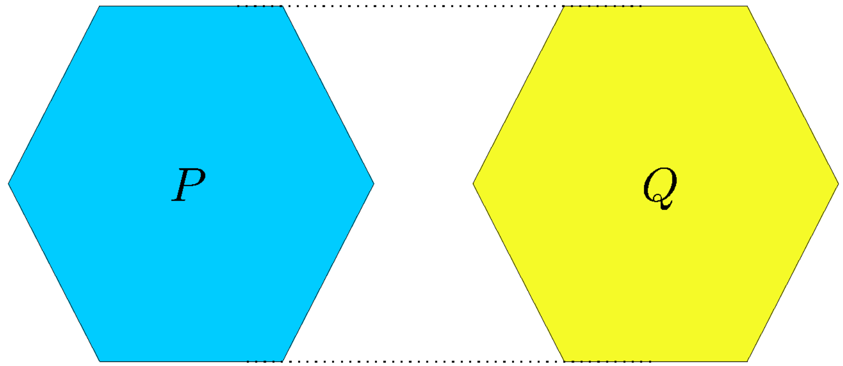
Theorem 3 *Let P and Q be VH-polytopes, $P \cup Q$ is convex* $\Rightarrow$ $\mathrm{conv}(V \cup W) = \mathrm{env}(P, Q)$, *moreover* $\mathrm{conv}(V \cup W) = \mathrm{env}(P, Q) = P \cup Q$.

The converse is not true:



Not useful for convexity recognition, a converse result can be proved under additional assumptions

VH-Polytopes



Theorem 4 *Let* $\bar{V} = \{v \in V : \ v \notin Q\}$, $\bar{W} = \{w \in W : \ w \notin P\}$. *If* $\bar{V} \cup \bar{W} \cup (V \cap W)$ *and* $\mathrm{env}(P, Q)$ *are minimal V- and H-representations, respectively, of the same polytope then* $P \cup Q$ *is convex, and* $P \cup Q = \mathrm{conv}(\bar{V} \cup \bar{W} \cup (V \cap W)) = \mathrm{env}(P, Q)$.

Proof. by contradiction.

The result can not be exploited for convexity recognition of the union of polyhedra, as checking coherence of given V- and H-representation might be a hard task

Algorithm for VH-Polyhedra

Assuming that P and Q are given by coherent VH-representations, an efficient algorithm can be proposed by modifying 2nd Algorithm for V-polytopes to exploit information coming from H-representations

In this case the algorithm computes the solution without solving any LP

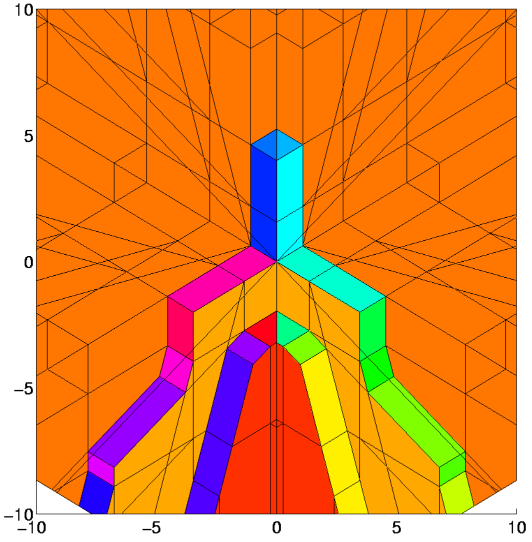
Complexity: $O(n_1 n_2 d(m_1 + m_2))$ strongly polynomial

Related Work (1): Reduction

Multiparametric Programming ammounts to solve for all x

$$\min_z \left\{ Rz + Qx \right\},$$
$$\text{subj. to } Gz \leq W + Kx$$

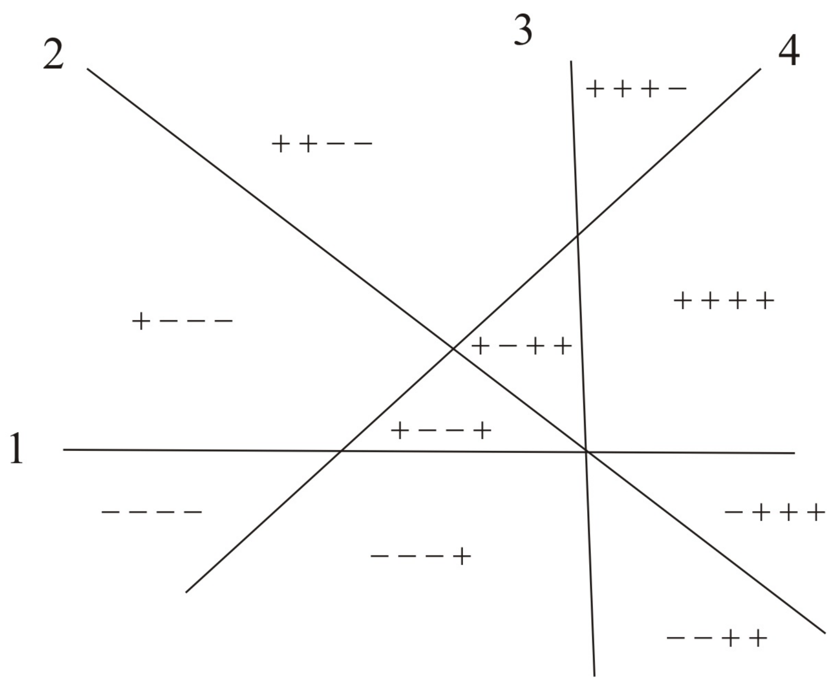
Optimal solution looks like



Problem: Find a "minimal" representation for the solution

Hyperplane Arrangements

Buck 1943,
Edelsbrunner 1987,
Fukuda 1996



Let $\mathcal{A}=\{H_i\}_{i=\{1,\dots,n\}}$, $H_i=\{x{:}a_ix\text{-}b_i=0\}$ be a collection of n hyperplanes in $\mathbb{R}^d$

Theorem Each polyhedral region (or cell) is associated to a sign marking

Theorem The total number of cells is bounded by Buck's formula

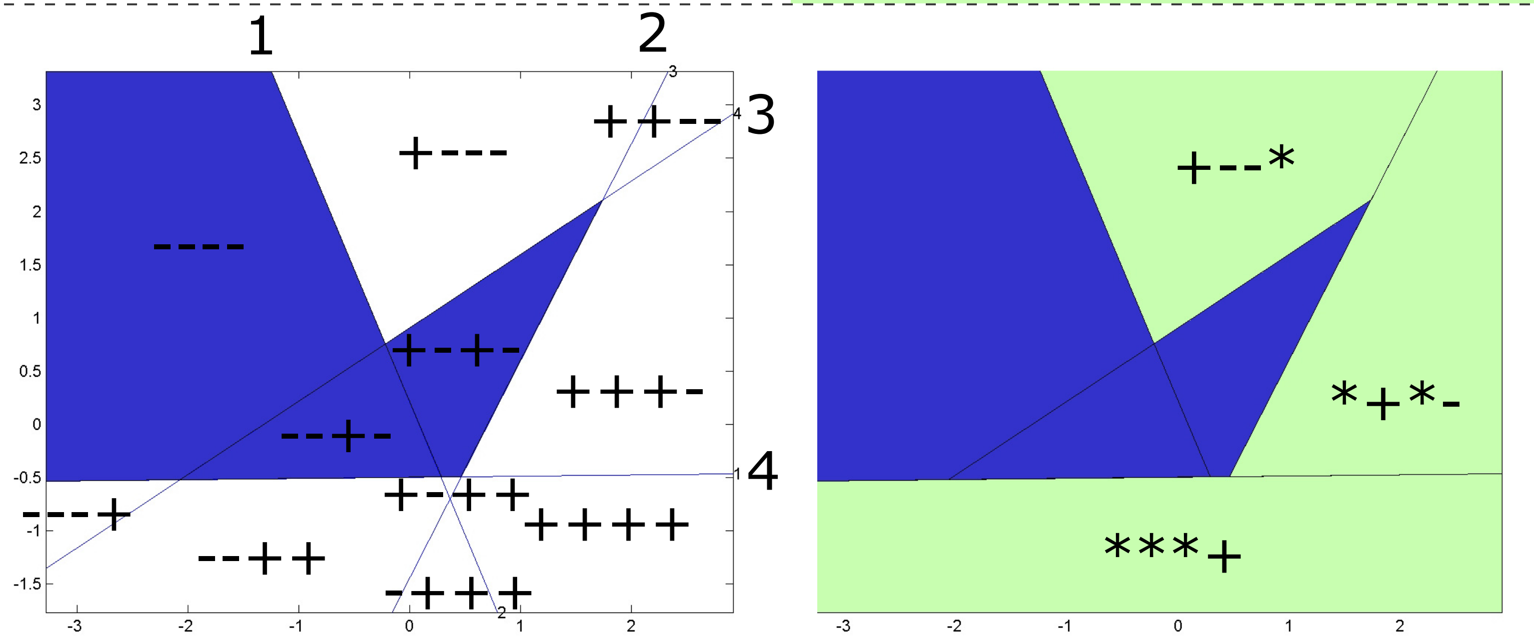
$$\#M \leq \sum_{i=0}^{d} \binom{n}{i}$$

Hyperplane Arrangements - Algorithms

There is an optimal algorithm for enumeration of hyperplane arrangements with time and space complexity $O(n^d)$ (Edelsbrunner '87)

There is reverse search algorithm (Fukuda '96,'01) for enumeration of hyperplane arrangements that runs in $O(n\ \mathrm{lp}(n,d)\ \#M)$ time and $O(n,d)$ space, where $\mathrm{lp}(n,d)$ is the complexity of solving a linear program with d variables and n constraints

Optimal Merging Of Cells - Idea



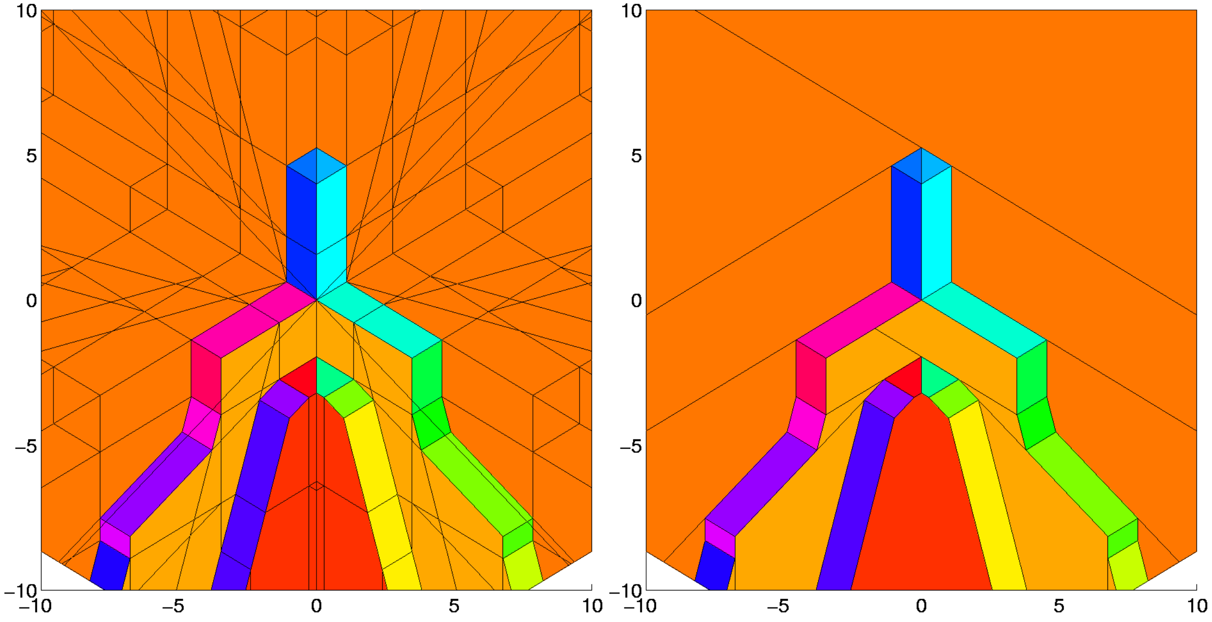
Markings allows easy merging of the region

- o Convexity recognition by marking comparison
- o Redundancy removal by marking comparison
- o Branch&Bound guarantees minimum by trying several combinations
- o Future research: Fast suboptimal algorithms

Optimal Merging

Geyer Torrisi '03

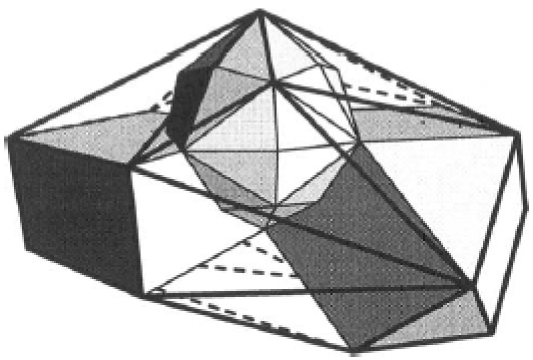
252 regions are reduced to 39 regions in 2' on Pentium IV 2.8 GHz machine



Related Work (2): Extended hull

Fukuda et al. 01

Compute the convex hull of the union of k H-polytopes



Conclusions

- We have provided:
  - Key theorems for characterizing the union of H-, V- and VH-polyhedra
  - Algorithms for computing the union of H-, V- and VH-polyhedra
- Similar work:
  - Efficient algorithms for adjacent H-polyhedra. (useful for multiparametric programming) (Geyer and Torrisi '03)
  - Convex hull of k H-polyhedra (Fukuda, Liebling and Lütolf '01)
- Open problems:
  - Generalization to k V-polyhedra

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