Inner and Outer Approximations of Polytopes Using Boxes

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Joint Work with
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Given an H-polytope $\mathcal{P} : \{x : Ax \leq b\}$ look for two collections $\mathcal{I}$ and $\mathcal{E}$ of adjacent boxes s. t.:

1. the union of all boxes in $\mathcal{I}$ is contained in $\mathcal{P}$
2. the union of all boxes in $\mathcal{E}$ contains $\mathcal{P}$

minimize the volume error and minimize the total number of boxes
Main Idea: Maximize the volume of the box subj. to. all the vertices are in $\mathcal{P}$

$\mathcal{B}(x, x + y) = \{z \in \mathbb{R}^d : x \leq z \leq x + y\}$; $\mathcal{P} = \{x \in \mathbb{R}^d : Ax \leq b\}$

$$\max_{x,y} \prod_{j \in D} y_j$$
subject to $$Ax + AV(S)y \leq b \quad (\forall S \subseteq D)$$
y > 0

where $D = \{1, \ldots, d\}$; $V(S) \in \{0,1\}^d$ is the incidence vector of $S \subseteq D$
Lemma 1 \( \text{The constraints } Ax + AV(S)y \leq b \ \forall S \subseteq D; y \geq 0 \) are equivalent to the set of constraints \( Ax + A^+y \leq b \), where \( A^+ \) is the positive part of \( A \).

**Proof** by lines, remembering that \( y > 0 \).

Lemma 2 \( \max_{x,y} \prod_{j \in D} y_j \) is equivalent to \( \max_{x,y} \sum_{j \in D} \ln y_j \). Therefore the problem is convex and polynomially solvable.
Single Inner Constrained Box

\[ \mathcal{B}(x, x + \lambda r) = \{z : z \leq z \leq x + \lambda r\}; \mathcal{P} = \{x : Ax \leq b\} \]

**Main Idea:** If the edges of the box are constrained, maximizing the volume amount to maximizing one edge

\[
\max_{x, \lambda} \lambda \\
\text{subject to } Ax + A^+ r \lambda \leq b,
\]

**Complexity:** \( O(lp(d + 1, m)) \)

**How to choose \( r \)?** \( e \) (1 vector), Inner Diameters, Outer Box
Single Greedy Inner Box

Assume that $0 \in \mathcal{P}$

How to find the max $\tau$ s.t. $\mathcal{B}(-\epsilon \tau, \epsilon \tau) \subseteq \mathcal{P}$?

$$\mathcal{B}(-\epsilon \tau, \epsilon \tau) = \{x \in \mathbb{R}^d: -\epsilon \tau \leq x \leq \epsilon \tau\}, \, \tau \in \mathbb{R};$$

$$\mathcal{P} = \{x \in \mathbb{R}^d: Ax \leq b\}, \, a_{ij} \text{ is the } j\text{-th element in the } i\text{-th row of } A$$

$$\tau(\mathcal{P}) = \max\{\tau: \mathcal{B}(-\tau \epsilon, \tau \epsilon) \subseteq \mathcal{P}\} = \min\{\tau_i: \, i = 1, \ldots, m\} \text{ where}$$

$$\tau_i = \begin{cases} \sum_{j \in D} b_i & \text{if } \sum_{j \in D} |a_{ij}| > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

because $z_i(\tau) = \max \left\{ \sum_{j \in D} a_{ij} x_j: \ x \in \mathcal{B} \right\}$
Single Greedy Inner Box

Main Idea: Starting from a point $x_0$ in $\mathcal{P}$, grow $B$ until it bridges one of the constraints of $\mathcal{P}$. Then, fix a vertex, remove the active constraints and continue, until all the vertices are fixed.

Complexity $O(md^2)$, $m = \#$ rows of $A$
Single Outer Box

\[ \mathcal{P} = \{x : Ax \leq b\} \]

**Main Idea:** Find the point \( u_j \) (\( l_j \)) in \( \mathcal{P} \) with the biggest (smallest) \( j \)-th coordinate

\[
\begin{align*}
    l_j &= \min\{x_j : Ax \leq b\} \\
    u_j &= \max\{x_j : Ax \leq b\}
\end{align*}
\]

**Complexity:** \( O(d \, \text{lp}(m, d)) \)
**Recursive Inner Approximation**

**Main Idea:** Partition $\mathcal{P} \setminus \mathcal{B}$ in $2d$ polyhedra and compute the inner approximation of the rests.

**Stopping Condition:** Prune a branch of the approximation if $\text{vol}(\mathcal{B}) < \epsilon$.

**Lemma 3** The total number of recursive calls is bounded by $2d \left\lceil \frac{\text{vol}(\mathcal{P})}{\epsilon} \right\rceil$.

**Theorem 1** Let $\mathcal{I}_\epsilon = \{\mathcal{B}_t\}_{t=1}^{S(\epsilon)}$ be the inner approximation of the polytope $\mathcal{P}$ for a given $\epsilon > 0$. Then,

$$\lim_{\epsilon \to 0} \bigcup_{t=1}^{S(\epsilon)} \mathcal{B}_t = \mathcal{P} \quad \text{a.e.}$$
Recursive Outer Approximation

Main Idea: Partition $\mathcal{P}$ in 2 polyhedra along the longest edge and compute the outer approximation of the two polyhedra

Stopping Condition: Prune a branch of the approximation if $\text{vol}(B) < \epsilon$

**Lemma 4** Let $V$ denote the volume of the minimum volume outer box of $\mathcal{P}$. The total number of boxes is bounded by $\left\lceil \frac{4V}{\epsilon} \right\rceil$.

**Theorem 1** Let $\mathcal{E}_\epsilon = \{B_t\}_{t=1}^{T(\epsilon)}$ be the outer approximation of the polytope $\mathcal{P}$ for a given $\epsilon > 0$. Then

$$\lim_{\epsilon \to 0} \bigcup_{t=1}^{T(\epsilon)} B_t \Rightarrow a.e. \mathcal{P}.$$
The multiple box outer approximation might generate too many polyhedra

Solution: Stop the approximation if $B$ is in the interior of $P$. I.e. $B \cap \delta P = \emptyset$

Alternative: Combine inner and outer approximation
Recursive Inner-Outer Approximation

**Main Idea:** First perform an inner approximation, then compute the outer approximation of the rests.

Computes in one shot both the inner and outer approximation.
Extension: Approximate Projections

**Main Idea:** First perform an inner approximation, then compute the outer approximation of the rests. Computes in one shot both the inner and outer approximation.
Conclusions

**Algorithms** to compute an inner and an outer approximation of a polytope

- Minimal *volume error* and *number* of boxes
- Alternative to the exact computation of the *projection*
- Good *performance*

**Open Question:** determine the projection of \( \mathcal{P} \) (or a polyhedral approximation) using the approximation

**Possible Extension:** Use arbitrary polytopes (i.e. octagons) as approximant shapes